Proceedings of the Indian Academy of Sciences

(Mathematical Sciences)

Subscription Rates

All countries except India $100

(Price includes AIR MAIL charges)

India .... Rs 150

Annual subscriptions are available for **Individuals** for India and abroad at the concessional rates of Rs. 75/- and $30 respectively.

All correspondence regarding subscription should be addressed to **The Circulation Department** of the Academy.

Editorial Office:

Indian Academy of Sciences, C V Raman Avenue,
P B No. 8005, Bangalore 560 080, India

Telephone: 334 2546
Telefax: 91-80-334 6094

Email: mathsci @ ias.ernet.in

The "Notes on the preparation of papers" are printed in the last issue of every volume.

R.N. 13242/57
Regd. No. KRN-A 41

ISSN 0253–4142

Vol. 108, No. 1, February 1998

Proceedings of the Indian Academy of Sciences

Mathematical Sciences

CONTENTS

Indexed in CURRENT CONTENTS

Edited and published by N Mukunda for the Indian Academy of Sciences, Bangalore 560 080.
Typeset and printed at Thomson Press (I) Ltd., Faridabad 121 007.

R.N. 13242/57
Regd. No. KRN-A 41

ISSN 0253–4142

Vol. 108, No. 2, June 1998

Proceedings of the Indian Academy of Sciences

Mathematical Sciences

CONTENTS

Indexed in CURRENT CONTENTS

Edited and published by N Mukunda for the Indian Academy of Sciences, Bangalore 560 080.
Typeset and printed at Thomson Press (I) Ltd., Faridabad 121 007.

STATEMENT ABOUT OWNERSHIP AND OTHER PARTICULARS OF

Proceedings of the Indian Academy of Sciences
Mathematical Sciences

1. Place of Publication	...	Bangalore
2. Periodicity of Publication	...	3 or 4 issues in a year
3. Printer's Name	...	N Mukunda Indian Academy of Sciences Bangalore 560 080
4. and 5. Publisher and Editor	...	N Mukunda
6. Nationality	...	Indian
7. Address	...	Indian Academy of Sciences P.B. No. 8005 Bangalore 560 080
8. Name and Address of the Owner	...	Indian Academy of Sciences Bangalore 560 080

I, N Mukunda, hereby declare that the particulars given above are true to the best of my knowledge.

N Mukunda
Signature of Publisher

Dated 1st March 1998

R.N. 13242/57
Regd. No. KRN-A 41

ISSN 0253–4142

Vol. 108, No. 3, October 1998

Proceedings of the Indian Academy of Sciences

Mathematical Sciences

CONTENTS

Indexed in CURRENT CONTENTS

Edited and published by N Mukunda for the Indian Academy of Sciences, Bangalore 560 080.
Typeset and printed at Thomson Press (I) Ltd., Faridabad 121 007.

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 1, February 1998, pp. 1–12.

Torus quotients of homogeneous spaces

S SENTHAMARAI KANNAN

SPIC Mathematical Institute, 92 G.N. Chetty Road, T. Nagar, Madras 600 017, India
E-mail: kannan@ssf.ernet.in

MS received 24 February 1997; revised 21 August 1997

Abstract. We study torus quotients of principal homogeneous spaces. We classify the Grassmannians for which semi-stable = stable and as an application we construct smooth projective varieties as torus quotients of certain homogeneous spaces. We prove the finiteness of the ring of T invariants of the homogeneous co-ordinate ring of the Grassmannian $G_{2,n}$ (n odd) over the ring generated by R_1, the first graded part of the ring of T invariants.

Keywords. Torus; Borel subgroups; Frobenius splitting; stable and semi-stable points.

1. Introduction

One of the classical problems in invariant theory is the study of binary quantics. The main object was to give an explicit description and study the geometric properties of SL_2 quotients of the projective space for a suitable choice of linearization.

The aim of this paper is to begin the study of a natural generalization of this classical question.

Let k be an algebraically closed field. Let G be a semisimple algebraic group over k, T a maximal torus of G, B a Borel subgroup of G containing T, N the normalizer of T in G, $W = N/T$, the Weyl group.

Consider the quotient variety $N\backslash\backslash G/B$. In fact the aim is to study more generally the variety $N\backslash\backslash G/Q$, where Q is any parabolic subgroup of G containing B.

In the case when $G = SL_n(k)$, the special linear group and Q is the maximal parabolic subgroup of $SL_n(k)$ associated to the simple root α_2, one knows that G/Q is the Grassmannian $G_{2,n}$ of two dimensional subspaces of an n dimensional vector space. One also has an isomorphism:

$$N\backslash\backslash(G/Q)^{ss}(L) = N\backslash\backslash(G_{2,n})^{ss}(L) \xrightarrow{\sim} SL_2\backslash\backslash(P(V))^{ss},$$

where V is the vector space of homogeneous polynomials of degree n in two variables and L is the line bundle associated to the fundamental weight ϖ_2, and the scheme $SL_2\backslash\backslash P(V)^{ss}$ is precisely the space of binary quantics (for example, see the proof of Theorem [1] and the proof of Theorem [4] of [CSS III]. We will also give an outline of the proof in the proof of corollary [3.10]).

More generally one has an isomorphism:

$$T\backslash\backslash(G/P)^{ss}(L) = T\backslash\backslash G^{ss}_{r,n}(L) \xrightarrow{\sim} SL_r\backslash\backslash(P^{r-1})^n,$$

where $G = SL_n(k)$, P is the maximal parabolic subgroup associated to α_r, $G_{r,n}$ is the Grassmannian of r dimensional subspaces of an n dimensional vector space and L the line bundle on $G/P = G_{r,n}$ associated to ϖ_r.

In this paper, we prove the following results:

(a) The varieties $T\backslash\backslash G/Q$ and $N\backslash\backslash G/Q$ are Frobenius split and as an application th vanishing Theorems for higher cohomologies of these varieties.
(b) As a part of result (a), we prove the vanishing of the higher cohomology groups fc the binary quantics.
(c) For the line bundle L on $G_{r,n}$ associated to the fundamental weight ϖ $(G_{r,n})^{ss}_T(L) = (G_{r,n})^{s}_T(L)$ if any only if r and n are coprime.
(d) Existence of smooth projective varieties as quotients of certain G/Q modul a maximal torus T(in the case of $G = SL_n$).
(e) For n odd, a partial result about R_1 generation of the graded ring $k[\widehat{G_{2,n}}]^T =$ $\bigoplus_{d\geqslant 0} R_d$.

The layout of this paper is as follows: Section 2 consists of notations, conventior and basic Theorems. In § (3), we prove the results (a), (b), (c) and (d), and in § (4), w prove the results (e) and (f).

2. Notations and conventions

Let k be an algebraically closed field. Let G be a semisimple algebraic group over k Ta maximal torus of G and B a Borel subgroup of G containing T.

Let N be the normalizer of T in G, $W = N/T$, the Weyl group. Let Φ be the set of a roots. Let Φ^+ and Φ^- be the sets of positive and negative roots respectively. Le $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ be the set of all simple roots, where l is the rank of G. Let s_i be th simple reflection with respect to α_i.

Let ϖ_i's be the fundamental weights.

Let $X(T)$ and $\Gamma(T)$ denote the sets of all characters and one parameter subgroups c T respectively. Denote the canonical bilinear form by

$$\langle , \rangle : X(T) \times \Gamma(T) \to Z.$$

Let $E = \Gamma(T) \otimes R$,

$$\overline{C(B)} = \{\lambda \in E : \langle \alpha, \lambda \rangle \geqslant 0 \text{ for all } \alpha \in \Delta\}.$$

Recall in the case of $G = SL_n$, G/P_r is the Grassmannian $G_{r,n}$ of r dimensional subspace of an n dimensional vector space, where P_r is the maximal parabolic subgrou associated to the simple root α_r. Define the character ε_i of T by $\varepsilon_i(t) = t_i$ for $t =$ dia $(t_1, t_2 \ldots t_n) \in T$.

Let $\hat{G}_{r,n}$ denote the cone over $G_{r,n}$ with respect to the line bundle given by the charac ter ϖ_r. Note that the action of an element t in Ton the vector $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r} \in \Lambda^r$ (V being the standard G module) is given by $t.e_{i_1} \wedge \ldots \wedge e_{i_r} = (\Pi^r_{j=1} t_{i_j}) \cdot e_{i_1} \wedge \ldots \wedge e$ hence on the co-ordinate ring $k[\hat{G}_{r,n}]$ is given by $t \cdot p_{i_1 i_2 \cdots i_r} = (\Pi^r_{j=1} t_{i_j})^{-1} \cdot p_{i_1 i_2 \cdots i_r}$ wher $p_{i_1 i_2 \cdots i_r}$ is the Plücker co-ordinate associated to $(i_1, i_2, \ldots, i_r)$ with $i_1 < i_2 < \ldots < i_r \leqslant$ Let $k[\hat{G}_{r,n}]^T$ denote the T invariants of the co-ordinate ring $k[\hat{G}_{r,n}]$.

Note also that if $(j_1, j_2, \ldots, j_r)$ is not in ascending order,

$$P_{j_1 j_2 \cdots j_r} = sgn(\tau) \cdot (p_{\tau(j_1) \cdots \tau(j_r)})$$

where τ is the permutation such that

$$1 \leqslant \tau(j_1) < \tau(j_2) < \ldots < \tau(j_r) \leqslant n.$$

The action of w, an element of W on $p_{i_1 \ldots i_r}$ is given by $w \cdot p_{i_1 \ldots i_r} = p_{w^{-1}(i_1) \ldots w^{-1}(i_r)}$. Denote $\tau \cdot p_{1,2,\ldots,r}$ by $p(\tau)$. For an r tuple $\mu = (i_1, i_2, \ldots, i_r)$, denote $p_{i_1, i_2, \ldots, i_r}$ by p_μ.

Now, coming back to the general semisimple algebraic group G, let w_0 denote the longest element of the Weyl group of G, ρ denote the half sum of all positive roots. The coset wB in G/B is denote by $e(w)$.

Let U denote the unipotent radical of B. When the characteristic p of the field k is positive we have the Frobenius map $F: \mathcal{O}_X \to \mathcal{O}_X$ defined by $F(f) = f^p$.

Denote the global sections of X with respect to the line bundle L on X by $H^0(X, L)$, and denote by $p(id)$ a lowest weight vector of the G module $H^0(G/B, L_\lambda)$ where L_λ is the line bundle associated to the dominant weight λ.

Let $G_{a,\alpha}$ denote the unipotent group associated to the root α. It is known that $U = \Pi_{\alpha > 0} G_{a,\alpha}$. Let X_α be the co-ordinate function of the affine line $G_{a,\alpha}$. Let $X(\tau)$ denote the Schubert variety defined by τ in W.

We use the notations for semistable points, stable points, numerical functions,... as in [GIT].

We recall some important results of [CSS I] here which will be used in § 3.

Lemma 2.1. *[CSS I] Let G be a semisimple algebraic group, T a maximal torus of G, B a Borel subgroup of G containing T, and $\overline{C(B)}$ be as in this section.*
(a) *Let L be the line bundle defined by the character $\chi \in X(T)$. Then if $x \in G/B$ is represented by bwB, $b \in B$, $w \in W$ represented by an element of N in the Bruhat decomposition of G and λ is a $1-PS$ of T which lies in $\overline{C(B)}$, we have*

$$\mu^L(x, \lambda) = -\langle w(\chi), \lambda \rangle.$$

(b) *Given any set S of finite number of non-trivial 1-PS λ of T, there is an ample line bundle L on G/B such that*

$$\mu^L(x, \lambda) \neq 0$$

for all $x \in G/B$, $\lambda \in S$.

3. Torus quotients

In this section we shall study the quotients of the homogeneous spaces under torus actions. In particular we give some natural criteria under which semi-stable points are actually stable. We also prove the Frobenius splitting property for these quotients.

PROPOSITION 3.1

Let G be a simply connected semisimple algebraic group, T a maximal torus of G, B a Borel subgroup of G containing T. Set $X = G/B$. Then there is a G linearized very ample line bundle L on X such that

$$X_T^{ss}(L) = X_T^s(L).$$

Proof. Choose a set S of finite number of non-trivial $1 - PS$ λ of $\overline{C(B)}$, that generates $\overline{C(B)}$ as a $R_{\geqslant 0}$ monoid. By Proposition [2.1 (b)], there is a very ample line bundle L on X such that

$$\mu^L(x, \lambda) \neq 0 \text{ for } \lambda \in S, x \in G/B. \qquad (*)$$

For $\lambda \in \overline{C(B)}$, write $\lambda = \sum_{\lambda_s \in S} a_s \lambda_s$, $a_s \geqslant 0$. By Lemma [2.1 (a)],

$$\begin{aligned}\mu^L(x,\lambda) &= -\left\langle w(\chi), \sum_s a_s \lambda s \right\rangle \\ &= -\left(\sum_s a_s \langle w(\chi), \lambda_s \rangle\right) \\ &= \sum_s a_s \mu^L(x, \lambda_s). \qquad (**)\end{aligned}$$

Now, let $x \in X_T^{ss}(L)$ be arbitrary. By Theorem [2.1] of [GIT], $\mu^L(x,\lambda) \geqslant 0$ for all 1–PS λ of T i.e, $\mu(wx, w\lambda w^{-1}) = \mu(x,\lambda) \geqslant 0$ for all 1–PS λ of T. So, $\mu(wx,\lambda) \geqslant 0$ for all $w \in W$, $\lambda \in C(B)$. Therefore by (*), we have $\mu^L(wx, \lambda_s) > 0$ for all $w \in W$ and $s \in S$.

By (**), we have

$$\mu^L(wx,\lambda) = \sum_s a_s \mu^L(wx, \lambda_s) \text{ for } \lambda \in \overline{C(B)}.$$

Hence for any nontrivial $1-\mathrm{PS}\lambda$ lying in $\overline{C(B)}$, $w \in W$,

$$\mu(wx,\lambda) > 0.$$

Since for any $1-\mathrm{PS}\,\lambda$ of T, there is a $w \in W$ such that $w\lambda w^{-1} \in \overline{C(B)}$, we have $\mu^L(x,\lambda) > 0$ for all nontrivial $1-\mathrm{PS}\,\lambda$ of T. Hence we have

$$X_T^{ss}(L) = X_T^{s}(L).$$

Lemma 3.2. *Let G be a semisimple algebraic group of adjoint type such that none of its simple factor is $\mathbb{P}SL_2(k)$. Let T be a maximal torus of G, B a Borel subgroup of G containing T. Let $X = G/B$. Then the set of points of X whose isotropy in T is non trivial has codimension at least two.*

Proof. Let D be an arbitrary prime divisor of X.

Case 1: $D \bigcap U{\cdot}e(w_0)$ is non empty. For any root α, set

$$Z(X_\alpha) = \{x = u{\cdot}e(w_0) \in U\,e(w_0)/X_\alpha(u) = 0\}.$$

Subcase 1: $D \bigcap U{\cdot}e(w_0) \neq Z(X_\alpha)$ for all $\alpha \in \Delta$. Choose a point $x \in D \bigcap U{\cdot}e(w_0)$ such that $X_\alpha(x) \neq 0$ for all $\alpha \in \Delta$. Then the isotropy T_x is contained in the intersection $\bigcap_{\alpha\in\Delta} \ker(\alpha) = (1)$. Hence the isotropy $T_x = (1)$ for some $x \in D \bigcap U{\cdot}e(w_0)$.

Subcase 2: $D \bigcap U{\cdot}e(w_0) = Z(X_{\alpha_i})$ for some $i \in \{1,2,\ldots,l\}$. Choose a point $x \in D \bigcap U{\cdot}e(w_0)$, $x = u{\cdot}e(w_0)$, $u \in U$ such that $X_\alpha(u) \neq 0$ for all $\alpha \neq \alpha_i$. It is easy to see that for any group G as above, and for any i, either $\alpha_{i-1} + \alpha_i$ or $\alpha_i + \alpha_{i+1}$ is a root of G. So, $T_x = T_u \subset \bigcap_{\alpha\neq\alpha_i} \ker(\alpha)$, $\bigcap_{\alpha\neq\alpha_i} \ker(\alpha) \subset \bigcap_{j\neq i} \ker(\alpha_j) \bigcap \ker(\alpha_{i-1} + \alpha_i) = (1)$, if $\alpha_{i-1} + \alpha_i$ is a root. Hence, $T_x = (1)$. The argument is similar if $\alpha_i + \alpha_{i+1}$ is a root.

Case 2: $D \bigcap U{\cdot}e(w_0)$ is empty. In this case D is a Schubert divisor $= X(\tau)$, where $\tau = w_0{\cdot}s_i$. Set

$$S(\tau) = \{\alpha \in \Phi^+/\tau(\alpha) \in \Phi^-\} = \Phi^+ - \{\alpha_i\}.$$

Then the unipotent group $U(\tau) = \Pi_{\alpha\neq\alpha_i, \alpha\in\Phi^+} G_{a,\alpha}$ is isomorphic to the big cell $U{\cdot}e(\tau)$ of $X(\tau)$.

Choose a point $u \in U(\tau)$ such that $X_\alpha(u) \neq 0$ for all $\alpha \neq \alpha_i$. Now, if $x = u{\cdot}e(\tau)$, then $T_x = T_u = (1)$, by proof as in the previous case.

Theorem 3.3. *Let $X = G_{r,n}$, $L = L_r$, the line bundle on X given by the weight ϖ_r. Let T be a maximal torus of $G = SL_n(k)$. Then*

$$X_T^{ss}(L) = X_T^{s}(L)$$

if and only if r and n are coprime.

Proof. $\Leftarrow$:

Assume that r and n are coprime. It is known that $X = G/P$, P the maximal parabolic subgroup associated to the root α_r. It is easy to see that $\overline{C(B)}$ is generated as a $R_{\geqslant 0}$ monoid by $\lambda_1, \ldots, \lambda_l$ where $\langle \alpha_j, \lambda_i \rangle = n \cdot \delta_{j,i}$, $\delta_{j,i}$ is the Kroneker number, $n = l + 1$.

Now, let $x \in BwP$ be an arbitrary point. Let $\pi: G/B \to G/P$ be the canonical projection. Choose a point $y \in BwB$ such that $\pi(y) = x$. By Proposition [2.1] of [CSS], we have

$$\mu^{\pi^*(L)}(y, \lambda) = \mu^L(x, \lambda). \tag{$*$}$$

By Lemma [2.1 (a)], we have

$$\mu^{\pi^*(L)}(y, \lambda) = -\langle w(\varpi_r), \lambda \rangle \tag{$**$}$$

for $\lambda \in \overline{C(B)}$.

Claim. $\mu^L(x, \lambda_s) \neq 0$ for all $x \in X$, and $s \in \{1, 2, \ldots, l\}$. From $(*)$ and $(**)$, we have

$$\mu^L(x, \lambda_s) = -\langle w(\varpi_r), \lambda_s \rangle = -\left\langle \varpi_r - \sum_{m_i \geqslant 0} m_i \alpha_i, \lambda_s \right\rangle.$$

Case 1: $r \leqslant s$. Since

$$\varpi_r = \frac{1}{l+1}\left(\sum_{j=1}^{r-1} j(l-r+1)\alpha_j + \sum_{j=r}^{l} r(l-j+1)\alpha_j \right)$$

we have

$$\langle \varpi_r, \lambda_s \rangle = \left\langle \frac{1}{l+1}(r(l-s+1)\alpha_s), \lambda_s \right\rangle = \frac{1}{l+1}(r(l-s+1)n = r(n-s)).$$

Hence,

$$\mu^L(x, \lambda_s) = -\left\langle \varpi_r - \sum_{i=1}^{n} m_i \alpha_i, \lambda_s \right\rangle = -(r(n-s) - nm_s) \neq 0$$

as r and n are copime.

Case 2: $r > s$

$$\begin{aligned}
\mu^L(x, \lambda_s) &= -\left\langle \varpi_r - \sum_i m_i \alpha_i, \lambda_s \right\rangle \\
&= -\left(\langle \varpi_r, \lambda_s \rangle - \sum_i m_i n \delta_{is} \right) \\
&= (-1/l+1)(s(l-r+1)\langle \alpha_s, \lambda_s \rangle - m_s n^2) \\
&= (-1/l+1)(s(l-r+1)n - m_s n^2) \\
&= -((n-r)s - nm_s) \neq 0
\end{aligned}$$

since $n - r$ and n are coprime.

Hence, for $\lambda\in\overline{C(B)}$, write $\lambda=\Sigma a_i\lambda_i,\ a_i\geqslant 0$.

$$\mu^L(x,\lambda)=-\left\langle w(\varpi_r),\sum_i a_i\lambda_i\right\rangle$$

$$=-\sum_s a_s\langle w(\varpi_r),\lambda_s\rangle$$

$$=\sum_s a_s\mu^L(x,\lambda_s). \qquad (**$$

Let $x\in X_T^{ss}$ be arbitrary. Therefore, $\mu^L(x,\lambda)\geqslant 0$ for all $\lambda\in\Gamma(T)$ by Theorem [2.1] [GIT].

Hence

$$\mu(wx,\lambda)=\mu(x,w^{-1}\lambda w)\geqslant 0$$

for all $\lambda\in\overline{C(B)}$ and $w\in W$. So, $\mu(w^{-1}x,\lambda_s)>0$ for all $s\in\{1,2,\ldots,l\}$ and $w\in W$ (sin $\mu(w^{-1}x,\lambda_s)\neq 0$ for all s and $w\in W$).

By (***), we have

$$\mu^L(w^{-1}x,\lambda)=\sum_{\lambda_s\in S} a_s\mu(w^{-1}x,\lambda_s)>0$$

for all nontrivial $1-\mathrm{PS}\ \lambda\in\overline{C(B)}$ and $w\in W$.

Since for any $\lambda\in\Gamma(T)$, there is a $w\in W$ such that $w\lambda w^{-1}\in\overline{C(B)}$. Therefore we ha $\mu^L(x,\lambda)>0$ for all non trivial $1-\mathrm{PS}\lambda$ of T. Hence by Theorem [2.1] of [GIT $x\in X_T^s(L)$.

Hence we have

$$X_T^{ss}(L)=X_T^s(L).$$

$:\Rightarrow$ Conversely suppose that r and n are not coprime. So write $n=q\cdot d$; $r=m\cdot d$ wi $d>1$. For $k\in\{1,2,\ldots,r\}$ write $k=(j-1)m+i$, where $j\in\{1,\ldots,d\}$ and $i\in\{1,2,\ldots,m$ Choose $w\in W$ such that $w^{-1}(k)=jq-m+i$ where $k=(j-1)m+i$, $k\in\{1,2,\ldots,r\}$. L $x\in BwP$ be such that $p(\tau)(x)\neq 0$ for all $\tau\leqslant w$. For $s\in\{1,2,\ldots,q\}$, define

$$\mu_s=([s],\ldots,[s+m-1],q+[s],\ldots,q+[s+m-1],\ldots,(d-1)q$$
$$+[s],\ldots,(d-1)q+[s+m-1]),$$

where $[a]$ is defined to be the unique integer in $\{1,2,\ldots,q-1\}$ such that $[a]=a$ m q for $a\neq q$ and $[q]=q$.

Therefore the weight of p_{μ_s} is equal to

$$-\left[\sum_{j=1}^{d}\sum_{i=s}^{s+m-1}\varepsilon_{(j-1)q+[i]}\right].$$

Hence the weight of the section $\sigma=\Pi_s p_{\mu_s}$ of the line bundle $L^{\otimes q}$ is equal to

$$m\left[-\left(\sum_{j=1}^{d}\sum_{s=1}^{q}\varepsilon_{(j-1)q+s}\right)\right]=-(m)\left(\sum_{i=1}^{n}\varepsilon_i\right)=0.$$

Therefore σ is a T invariant section of $L^{\otimes q}$ such that $\sigma(x)\neq 0$. Hence x is semistab

Claim 2: x is not stable.

For, $\mu^L(\dot{x}, \lambda_q) = -\langle w(\varpi_r), \lambda_q\rangle$. Now,

$$w(\varpi_r) = w\left(\sum_{i=1}^{r} \varepsilon_i - (r/l+1)\left(\sum_{i=1}^{n} \varepsilon_i\right)\right).$$

So,

$$\langle w(\varpi_r), \lambda_q\rangle = \left\langle \sum_{k=1}^{r} \varepsilon_{w^{-1}(k)}, \lambda_q \right\rangle$$

since $\langle \sum_{i=1}^{n} \varepsilon_i, \lambda_q\rangle = 0$. i.e,

$$\langle w(\varpi_r), \lambda_q\rangle = \left\langle \sum_{j=1}^{d}\sum_{i=1}^{m} \varepsilon_{jq-m+i}, \lambda_q \right\rangle$$

$$= \left\langle \sum_{i=1}^{m} \varepsilon_{q-m+i}, \lambda_q \right\rangle + \left\langle \sum_{j=2}^{d}\sum_{i=1}^{m} \varepsilon_{jq-m+i}, \lambda_q \right\rangle$$

$$= m(n-q) - (d-1)mq = 0,$$

since $n = qd$, i.e, $\mu^L(x, \lambda_q) = 0$. Hence x is not stable.

COROLLARY 3.4

Let $G = SL_n$, T a maximal torus of G, B a Borel subgroup of G containing T. Let $Q = \bigcap_{\{j\in J\}} P_j$ be a parabolic subgroup of G containing B such that there is an $r\in J$ with r and n being coprime. Set $Z = G/Q$. Then there is a very ample G linearized line bundle N on Z such that $Z_T^{ss}(N) = Z_T^{s}(N)$.

Proof. Let $X = G_{r,n}$ where $r\in J$ be such that r and n are coprime. Let L be the line bundle on X associated to the weight ϖ_r. Let M be the line bundle on Z associated to the weight $\sum_{i\in J-\{r\}} \varpi_i$. Let $N = aL + bM$ be a line bundle on Z so that "b/a is sufficiently small" as in Proposition [5.1] of [CSS I]. Then by Proposition [5.1] of [CSS I],

$$Z_T^{ss}(N) = Z_T^{s}(N).$$

Lemma 3.5. *Let $G = SL_n$, T, B, Q, Z and N be as in Corollary* (3.4). *Then $T\backslash\backslash(Z_T^{ss}(N))$ is smooth.*

Proof. By Corollary (3.4), there is a very ample G linearized line bundle N on Z such that $Z_T^{ss}(N) = Z_T^{s}(N)$.

Claim. For $z\in Z^s$, the isotropy of z in T is the center of the group G.

Proof of Claim. Let $z\in Z^s$. Then T_z is finite i.e, $T\bigcap gQg^{-1}$ is finite, where $z = gQ$. Then $T\bigcap S$ is finite for every maximal torus S of gQg^{-1}.

Since G is reductive for any maximal torus S of G, S must be the centralizer $C_G(s)$ for some s in S. Therefore for any maximal torus S of gQg^{-1}, we have

$$T\bigcap S = \{t = \mathrm{diag}(t_1,\ldots,t_n)\in T\colon t_i t_j^{-1} = 1 \text{ if } s_{ij} \neq 0 \text{ or } s_{ij} \neq 0\}$$

is finite (where $s = (s_{ij})\in S$ is such that $S = C_G(s)$).

Therefore, the roots

$$\{\varepsilon_i - \varepsilon_j : s_{ij} \neq 0 \text{ or } s_{ji} \neq 0\}$$

generate the root lattice.

Hence $T \bigcap S$ is the center of G. Since any semisimple element of gQg^{-1} lies in so maximal torus S of gQg^{-1}, we have $T_z = T \bigcap gQg^{-1} = Z(G)$.

Hence the claim.

Now, if we set $\bar{G} = PSL_n = G/Z(G)$, $\bar{T} = T/Z(G)$, $\bar{B} = B/Z(G)$, $\bar{Q} = Q/Z(G)$.

Then $Z = \bar{G}/\bar{Q}$ and $Z^{ss}_{\bar{T}}(N) = Z^{s}_{\bar{T}}(N)$.

Therefore the isotropy $\bar{T}_z$ of z in Z^s is trivial. Hence by Proposition [0.9] of [GI $T\backslash\backslash Z^{ss}(N)$ is smooth.

Lemma 3.6. *Let $r \neq 2$ or $n \neq 4$. Let $G = SL_n$, T a maximal torus of G, B a Borel subgro of G containing T. Let P be the maximal parabolic subgroup of G associated to the sim root α_r. Let $X = G_{r,n} = G/P$. Let L be the line bundle associated to the fundamental wei ϖ_r. Then* $\operatorname{codim}(X^{ss}_T(L) - X^{s}_T(L)) \geqslant 2$.

Proof. By Theorem (3.3), without loss of generality we can assume that $1 < r < n -$ $(= l)$ (since if $r = 1$ or $n - 1$, then $X^{ss}(L) - X^{s}(L)$ is empty set and therefore trivially t Lemma is true).

It is enough to prove that for any prime divisor D of X, $D \bigcap X^s$ is nonempty. Let D any prime divisor of X. Set $U = B{\cdot}e(w_0) = B{\cdot}(w_0P) \subset X$, the big cell.

Case 1: Assume that $w{\cdot}U$ meets D for every $w \in W$. Since D is irreducible and W is fini the intersection

$$\left(\bigcap_{w \in W} wU\right) \bigcap D$$

is nonempty. Now, choose a point $x \in (\bigcap_{w \in W} wU) \bigcap D$. Let λ be a nontrivial o parameter subgroup of T. Choose a $w \in W$ such that $w\lambda w^{-1} \in \overline{C(B)}$.

Since $wx \in U$,

$$\mu(x, \lambda) = \mu(wx, w\lambda w^{-1}) = -\langle w_0(\varpi_r), w\lambda w^{-1}\rangle = \langle \varpi_{n-r}, w\lambda w^{-1}\rangle.$$

Since $w\lambda w^{-1} \in \overline{C(B)}$, write $w\lambda w^{-1} = \sum_{a_s \geqslant 0} a_s \lambda_s$. So,

$$\mu(x, \lambda) = \sum_{s < n-r} a_s \langle \varpi_{n-r}, \lambda_s\rangle + \sum_{s \geqslant n-r} a_s \langle \varpi_{n-r}, \lambda_s\rangle$$

$$= \sum_{s < n-r} a_s(sr) + \sum_{s \geqslant n-r} a_s(n-r)(n-s).$$

Since λ is a nontrivial $1-\mathrm{PS}$ of T, there is a $s \in \{1, 2, \ldots, l\}$ such that $a_s > 0$, hen $\mu(x, \lambda) > 0$. Hence by Theorem [2.1] of [GIT], $x \in D \bigcap X^s$.

Case 2: Assume that there is a $w \in W$ such that wU does not meet D. Therefore we ha $w^{-1}D = X(\tau)$, where $\tau = w_0 \cdot s_r$. Choose $x \in X(\tau)$ such that

$$\mathrm{p}(\sigma)(x) \neq 0 \quad \text{for all} \quad \sigma \leqslant \tau.$$

Claim: x is stable. By Lemma [2.1(a)],

$$\mu(x, \lambda_s) = -\langle \tau(\varpi_r), \lambda_s \rangle = -\langle w_0(\varpi_r - \alpha_r), \lambda_s \rangle = \langle \varpi_{n-r} - \alpha_{n-r}, \lambda_s \rangle. \qquad (*)$$

For $s \neq n-r$,

$$\mu(x, \lambda_s) = \langle \varpi_{n-r}, \lambda_s \rangle > 0$$

by proof as in Theorem [3.3].

For $s = n-r$,

$$\mu(x, \lambda) = \langle \varpi_{n-r}, \lambda_s \rangle - n\delta_{n-r,s} = (n-r)(n-s) - n = nr - r^2 - n > 0$$

since $1 < r < n-1$ and $r \neq 2$ or $n \neq 4$. Therefore $\mu(x, \lambda) > 0$ for all 1–PS λ of T lying in the closure of the Weyl chamber $\overline{C(B)}$.

Now, let λ be a nontrivial 1–PSλ of T. Choose a $\sigma \in W$ so that

$$\lambda_1 = \sigma\lambda\sigma^{-1} \in \overline{C(B)}.$$

Since the weight of $p(w_0) = \varpi_{n-r}$, weight of $p(\tau) = \varpi_{n-r} - \alpha_{n-r}$, we have weight of $p(\sigma^{-1}\tau) \neq$ weight of $p(\sigma^{-1}w_0)$.

Therefore since the only extremal section vanishing at x is $p(w_0)$, we have either $p(\tau)(\sigma x) \neq 0$ or $p(w_0)(\sigma x) \neq 0$. So,

$$\mu(x, \lambda) = \mu(\sigma x, \sigma\lambda\sigma^{-1}) = \mu(\sigma x, \lambda_1) \geqslant -\langle \tau(\varpi_r), \lambda_1 \rangle = \mu(x, \lambda_1)$$

which is positive

Hence we have $x \in D \bigcap X^s$. Hence the Lemma.

Theorem 3.7. *Let G be a simply connected semisimple algebraic group. Let T be a maximal torus of G. Let B be a Borel sub group of G containing T. Let $X = G/B$. Let L be any ample line bundle on X. Then the good quotient $Y = T\backslash\backslash X^{ss}_T(L)$ is Frobenius split.*

Proof. By Theorem [2] (p. 38) [M-R], X is Frobenius split. Let V be a T invariant affine open subset of X. Let R be the coordinate ring of V. Let $\phi\colon F*R \to R$ be a R linear map such that $\phi \circ F = id_R$.

Let $\pi\colon R \to R^T$ be the Reynolds operator. Define $\tilde{\phi}\colon F*R^T \to R^T$ by $\tilde{\phi}(f) = \pi \circ \phi(f)$ for $f \in F*R^T$. It is easy to see that $\tilde{\phi}$ is R^T linear and $\tilde{\phi} \circ F = id_{R^T}$. Hence Y is Frobenius split.

COROLLARY 3.8

Let G be a simply connected semisimple algebraic group. Let T, B, X, L, Y be as in previous Theorem. Let N be the normalizer of T in G. Let W be the Weyl group. Then if p the characteristic of the field is bigger than the order of the group W, then the good quotient $Z = N\backslash\backslash X^{ss}_N(L)$ is Frobenius split.

Proof. Since $Z = W\backslash\backslash Y^{ss}_W$, Y is Frobenius split and W is linearly reductive (since p the characteristic of the field is bigger than the order of the Weyl group W) by proof as in previous Theorem Z is Frobenius split.

Theorem 3.9. *Let G be a simply connected semisimple algebraic group. Let T, B be as in Theorem* [3.7]. *Let Q be any parabolic subgroup of G containing B. Let $X = G/Q$, L be*

any very ample line bundle on X. Then we have

1. *The good quotient $Y = T\backslash\backslash X_T^{ss}(L)$ is Frobenius split.*
2. *If p the characteristic of the field is bigger than the order of the Weyl group W, the good quotient $Z = N\backslash\backslash X_N^{ss}(L)$ is Frobenius split.*

Proof. It is known that G/Q is Frobenius split (for a proof, see [M-R]). So using the Reynolds operator (as in Theorem [3.7] and corollary [3.8]) we can prove (1) as well as (2).

COROLLARY 3.10

Let V be the vector space of homogeneous polynomials of degree n in two varibales. Then for the natural action of SL_2 on the projective space $P(V)$ the good quotient $SL_2\backslash\backslash P(V)^{ss}$ is Frobenius split.

Proof. Let $M(2, n)$ denote space of $2 \times n$ matrices with entries in k. The morphism $\phi: M(2, n) \to V$ defined by

$$\phi\left(\begin{pmatrix} a_1 & a_2 & \ldots a_n \\ b_1 & b_2 & \ldots b_n \end{pmatrix}\right) = \pi_{i=1}^n (a_i X + b_i Y)$$

gives an isomorphism between the varieties $(P(M(2, n)))^{ss}//N$ and $P(V)$. On the other hand, the morphism $\chi: M(2, n) \to \Lambda^2(k^n)$ defined by

$$\chi\left(\begin{pmatrix} a_1 & a_2 & \ldots a_n \\ b_1 & b_2 & \ldots b_n \end{pmatrix}\right) = \left(\det\left(\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}\right)\right) i<j$$

gives an isomorphism between the varieties $SL_2\backslash\backslash(P(M(2, n)))^{ss}$ and the Grassmannian $G_{2,n}$. Hence the variety $N\backslash\backslash G_{2,n}^{ss}$ is isomorphic to

$$SL_2\backslash\backslash(P(M(2, n)))^{ss}//N)^{ss}$$

via χ and the latter is isomorphic to $SL_2\backslash\backslash(P(V))^{ss}$ via ϕ. Thus we have established an isomorphism:

$$N\backslash\backslash(G/Q)^{ss}(L) \xrightarrow{\sim} SL_2\backslash\backslash P(V)^{ss}$$

where $G = SL_n$, Q is the maximal parabolic subgroup associated to α_2 and L is the line bundle on G/Q associated to the fundamental weight ϖ_2. From Theorem [3.9] it is easy to see that $SL_2\backslash\backslash P(V)^{ss}$ is Frobenius split.

COROLLARY 3.11

Let V be the vector space of homogeneous polynomials of degree n in two variables. Then for any ample line bundle L on the good quotient $Z = SL_2\backslash\backslash P(V)^{ss}$ the cohomologies $H^i(Z, L) = 0$ for $i > 0$.

Proof. It is known that for any normal Frobenius split projective variety X, and for any ample line bundle L, $H^i(X, L) = 0$ for all $i > 0$. (For a proof, see [M-R].) Hence by Corollary (4.10), $H^i(Z, L) = 0$ for all $i > 0$.

4. R_1 generation

In this section we prove a partial theorem of the R_1 generation property for the homogeneous coordinate ring of $T\backslash\backslash G^{ss}_{2,n}(L)$ for odd n, where L is the line bundle on $G_{2,n}$ associated to the fundamental weight ϖ_2.

Let $G = SL_n$, T a maximal torus of G. Let B be a Borel subgroup of G containing T. Let P be the maximal parabolic subgroup associated to the simple root α_2. Let $X = G_{2,n} = G/P$, let L be the line bundle associated to the fundamental weight ϖ_2. Let X denote the cone over X with respect to the line bundle L.

Let $k[\hat{X}]$ denote the co-ordinate ring of $\hat{X}$. Let $\{p_{ij}: i<j, i,j\in\{1,2,\ldots,n\}\}$ denote the set of Plücker co-ordinates.

Recall from § 1, for $t = \mathrm{diag}(t_1, t_2,\ldots,t_n)\in T$, the action of t on p_{ij} is given by

$$t\cdot p_{ij} = (t_i t_j)^{-1} p_{ij}.$$

An element $f\in k[\hat{X}]$ is T invariant if and only if $f = \sum_i a_i M_i$, $a_i\in k$, each M_i in the sum is a T invariant monomial in p_{ij}'s. (For a proof: Since the standard monomials [cf. [CSS II]] form a basis for the k vector space $k[\hat{X}]$ and they are all T weight vectors, any $f\in k[\hat{X}]^T$ can be written as $f = \sum_\chi a_\chi M_\chi$, $a_\chi\in k$ and M_χ's are standard monomials in the $p_{i,j}$'s of T weight χ, forcing that the weight χ is zero whenever $a_\chi \neq 0$.)

Now, for a monomial $f = \prod_{i<j} p_{ij}^{m_{ij}}$ set $m_{ij} = m_{ji}$ for $i>j$. With these additional symbols $\{m_{ij}: i>j\}$ it is easy to see that a monomial $f = \prod_{i<j} p_{ij}^{m_{ij}}$ is T invariant if and only if $\sum_{k\neq i} m_{ik} = \sum_{k\neq j} m_{jk}$ for every $i,j\in\{1,\ldots,n\}$.

Write $k[\hat{X}] = A = \oplus_{d\geq 0} A_d$, where A_d is the space of homogeneous elements of degree d in the p_{ij}'s. For n odd, it is easy to see that

$$A_d^T \neq 0$$

if and only if n divides d. Since the action of T on X is linear with respect to L, the ring of T invariants $R = k[\hat{X}]^T$ is a graded ring, say $R = \oplus_{d\geq 0} R_d$.

Let S be the subring of R generated by the vector space R_1 of R.

Let M denote the monoid of $n\times n$ symmetric matrices with non negative integer entries and whose diagonal is zero.

Let

$$M_R = \{(m_{ij})\in M : \sum_{j\neq i} m_{ij} = 2d \quad \text{for some} \quad d\in Z_{\geq 0}, \quad \text{for all} \quad i\in\{1,2,\ldots n\}\}.$$

Let

$$M_S = Z_{\geq 0}\,\mathrm{span}\{(m_{ij})\in M : \sum_{j\neq i} m_{ij} = 2 \quad \text{for all} \quad i\in\{1,\ldots,n\}\}.$$

It is easy to see that the monomials $\{\prod_{i<j} p_{ij}^{m_{ij}} : (m_{ij})\in M_R\}$ generate the vector space R over k. Also it is easy to see that $\{\prod_{i<j} p_{ij}^{m_{ij}} : (m_{ij})\in M_S\}$ generate the vector space S over k.

DEFINITION 4.1

A square matrix $v = (m_{ij})$ is called 'doubly stochastic' if $\sum_{i=1}^n m_{ij} = 1$ for every $j\in\{1,2,\ldots,n\}$, $\sum_{j=1}^n m_{ij} = 1$ for every $i\in\{1,2,\ldots,n\}$, and m_{ij}'s are non negative real numbers.

DEFINITION 4.2

A permutation matrix is a $\{0,1\}$ square matrix with exactly one 1 in each row and each column.

Theorem 4.3. [cf [S] $p\cdot 108$]. *A matrix v is a 'doubly stochastic' matrix if and only if v is a convex combination of permutation matrices.*

Theorem 4.4. *If n is odd, the ring of T invariants R is a finite module over S of degree 2^r for some r.*

Proof. Since T is a reductive group, R is a finitely generated k algebra. Choose a finite set $\{f_s : s \in \{1, 2, \ldots, N\}\}$ of monomials generating R.

Write $f_s = \prod_{i<j} p_{ij}^{m_{ij}}$ for $s \in \{1, \ldots, N\}$. Therefore the matrix $v = (m_{ij}) \in M_R$. So, $1/2d$ times v is a 'doubly stochastic' matrix for some positive integer d. By Theorem (5.4), $(1/2d)\cdot v$ is a convex combination of permutation matrices. Hence v is a sum of permutation matrices say equal to $\sum_i \sigma_i$ (not neccessarily distinct).

Since v is symmetric,

$$2v = v + v^t = \sum_i (\sigma_i + \sigma_i^t) \in M_S.$$

Therefore $f_s^2 \in S$. Hence the Theorem.

Acknowledgements

The author wishes to thank Prof. C S Seshadri for his encouragement and his help and to Profs T R Ramadas, V B Mehta, S Raghavan and Kashiwara and also to V Balaji, P A Vishwanath, P Sankaran and K N Raghavan for helpful discussions.

References

[M-R] Mehta V B and Ramanathan A, Frobenius splitting and cohomology vanishing for Schubert varieties, *Ann. Math.* **122** (1985) 27–40

[GIT] Mumford D, Fogarty J and Kirwan F, *Geometric Invariant Theory* (third edition) (New York: Springer-Verlag, Berlin, Heidelberg)

[S] Schrijver A, *Theory of Linear and Integer Programming* (John Wiley and Sons Ltd.) (1986)

[CSS I] Seshadri C S, Quotient spaces modulo reductive algebraic groups, *Ann. Math.* **95** (1972) 511–556

[CSS II] Seshadri C S, *Introduction to the theory of standard monomials*, Brandeis Lecture Notes 4, June 1985

[CSS III] Seshadri C S, *Mumford's conjecture for GL(2) and applications.* International Colloquim on Algebraic Geometry, Bombay, 16–23 January (1968)

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 1, February 1998, pp. 13–21.

Finite dimensional imbeddings of harmonic spaces

K RAMACHANDRAN and A RANJAN

Department of Mathematics, Indian Institute of Technology, Powai, Mumbai 400 076, India
Email: {kram, aranjan}@math.iitb.ernet.in

MS received 11 October 1996; revised 30 September 1997

Abstract. For a noncompact harmonic manifold M we establish finite dimensionality of the eigensubspaces V_λ generated by radial eigenfunctions of the form $\cosh r + c$. As a consequence, for such harmonic manifolds, we give an isometric imbedding of M into (V_λ, B), where B is a nondegenerate symmetric bilinear indefinite form on V_λ (analogous to the imbedding of the real hyperbolic space H^n into $\mathbb{R}^{n+1}$ with the indefinite form $Q(x,x) = -x_0^2 + \Sigma x_i^2$). This imbedding is minimal in a 'sphere' in (V_λ, B). Finally we give certain conditions under which M is symmetric.

Keywords. Harmonic manifolds; eigen spaces; imbeddings; symmetric spaces.

1. Introduction

A Riemannian manifold (M, g) is said to be harmonic if its volume density function $\omega_m = \sqrt{|\det(g_{ij})|}$ is a spherically symmetric function around m. In polar coordinates this density can be written as

$$\theta_m = r_m^{d-1}\,\omega_m,$$

where $r_m(n) = r(m, n)$ is the geodesic distance between m and n. Thus (M, g) is harmonic if θ_m is a radial function around m (see [8] or [1] for details).

Let (M, g) be a harmonic manifold and let Δ be the Laplacian on M. Consider a nonzero eigenvalue λ and the corresponding eigenspace V_λ. If M is compact, V_λ is known to be finite dimensional. In this case Besse [1] constructed isometric immersion of M into V_λ. Moreover the image of M in this case is a minimal submanifold of some sphere in V_λ. This immersion was crucially used by Szabo [10] in his proof of the Lichnerowicz conjecture, which asserts that *harmonic manifolds are symmetric*. Szabo proved the conjecture for compact harmonic manifolds with finite fundamental group. In his paper he generalized Besse's imbedding theorem to give isometric immersions of any harmonic manifold M into $L^2(M, g)$, an infinite dimensional space. In particular he recovers Besse's imbedding for compact harmonic manifold.

The eigenspaces V_λ of noncompact manifolds, in particular of noncompact harmonic manifolds, need not be finite dimensional. Nevertheless the problem of imbedding M into a finite dimensional vector space albeit with an indefinite metric is important and has not been addressed (analogous to the imbedding of the real hyperbolic space H^n into $\mathbb{R}^{n+1}$ with the indefinite metric $Q(x, x) = -x_0^2 + \Sigma x_i^2$). We prove a partial result in this direction. Before we state our result let us recall that the action of the Laplacian on radial functions f is

$$f''(r) + \frac{\theta'(r)}{\theta(r)} f' + \lambda f = 0,$$

where $\theta(r)$ is the density function of M. We now state our theorem.

Theorem 1.1. *Let* (M^d, g) *be a noncompact harmonic manifold with density function* $\theta(r)$. *Let* (f, λ) *be a solution of the equation*

$$f''(r) + \frac{\theta'(r)}{\theta(r)} f' + \lambda f = 0 \tag{1.1}$$

with $\lambda \neq 0$. *Assume that* f *is of the form*

$$f(r) = \cosh r + D$$

where D *is a constant. Let* V_λ *be the eigensubspace generated by radial eigenfunctions. Then*

1. V_λ *is finite dimensional and*
2. *there exists a symmetric nondegenerate bilinear form* B *on* V_λ *and an isometric imbedding of* M *into* $(V_\lambda, -B)$. *Moreover the image of* M *is a minimal submanifold of a 'sphere' in* V_λ.

Moreover in this case the volume density is

$$\theta(r) = 2^{d-1}\left(\sinh\frac{r}{2}\right)^{d-1}\left(\cosh\frac{r}{2}\right)^{d-1-4/3(k+d-1)}$$

We can also calculate λ *and* D *in terms of the Ricci curvature. They are given by the formulas*

$$D = \frac{k+d-1}{\frac{d}{2}+1-k}$$

and

$$\lambda = -\frac{2}{3}\left(\frac{d}{2}+1-k\right)$$

where Ricci $(M) = k$. *Finally* $f(r) = \cosh r + D$ *has no zeroes on* M.

Note. $D + 1 > 0$ and V_λ is not the full eigenspace. Moreover the fact that the density function is

$$\theta(r) = 2^{d-1}\left(\sinh\frac{r}{2}\right)^{d-1}\left(\cosh\frac{r}{2}\right)^{d-1-(4/3)(k+d-1)}$$

is equivalent to the fact that there exists an eigenfunction of the form

$$f(r) = \cosh r + D,$$

where

$$D = \frac{k+d-1}{(d/2)+1-k}.$$

In the next section we prove our main theorem. In the last section we prove symmetry of M under some assumptions.

2. The imbedding of M into $(V_\lambda, -B)$

Before proving our main theorem we first show that all known examples of noncompact harmonic spaces satisfy the hypothesis of our theorem. Thus all known noncompact

harmonic manifolds admit the desired imbedding. Nevertheless we strongly believe that all harmonic manifolds admit the desired imbedding. Notice that all the noncompact rank one symmetric spaces are NA spaces [3, 4].

Examples

1. *Real hyperbolic space* $\mathbb{R}H^d$. We know that the density function of the real hyperbolic space is

$$\theta(r) = \sinh^{d-1} r.$$

Thus the equation we wish to solve is

$$f''(r) + (d-1)\coth r f'(r) + \lambda f(r) = 0.$$

Put $f(r) = \cosh r$ in the above equation to get

$$f''(r) + (d-1)\coth r f'(r) = d\cosh r.$$

Hence $(\cosh r, -d)$ is the required solution.

2. *Complex hyperbolic space* $\mathbb{C}H^d$. In this case the density function is

$$\theta(r) = 2^{2d-1}\left(\sinh\frac{r}{2}\right)^{2d-1}\cosh\frac{r}{2}.$$

The function $f(r) = \cosh r + (d-1)/(d+1)$ satisfies the equation

$$f''(r) + \frac{\theta'(r)}{\theta(r)} f'(r) - (d+1)f = 0.$$

Thus $(\cosh r + ((d-1)/(d+1)), -(d+1))$ is the required solution.

3. *The NA spaces.* Recall that the density of the NA spaces ([3]) is of the form

$$\theta(r) = 2^{m+l}\left(\sinh\frac{r}{2}\right)^{m+l}\left(\cosh\frac{r}{2}\right)^{l}.$$

Hence

$$\frac{\theta'(r)}{\theta(r)} = \frac{1}{2}\left\{(m+l)\coth\frac{r}{2} + l\tanh\frac{r}{2}\right\}.$$

Assume $f(r) = \cosh r + D$, then

$$f''(r) + \frac{\theta'(r)}{\theta(r)} f'(r) = \frac{m+2l+2}{2}\cosh r + \frac{m}{2}.$$

Thus $(\cosh r + (m/(m+2l+2)), -(m+2l+2)/2)$ is the required solution. Let us make a simple but useful observation.

Observation. Let V be a vector space, $S \subseteq V$ a set such that $V = \operatorname{span} S$. Let $\tilde{B}: S \times V \to R$ be a map such that

1. $\tilde{B}$ is linear in the second variable, i.e., $\tilde{B}(s, .)$ is linear on V for all s in S.
2. $\tilde{B}(s, t) = \tilde{B}(t, s)$ for all $s, t \in S$.

Then there exists a unique bilinear form $B: V \times V \to R$ which extends $\tilde{B}$.

Proof. Define

$$B\left(\sum_i a_i s_i, v\right) = \sum_i a_i \tilde{B}(s_i, v).$$

Proof of Theorem 1.1. Let m be any point of M and let B_m be the unit ball around m. Let $C^\infty(B_m)$ be the space of smooth functions on B_m with the sup norm.

Consider the restriction map

$$\Psi: V_\lambda \to C^\infty(B_m); \quad f \mapsto f|_{B_m}.$$

Due to analyticity Ψ is injective. We will show that $\operatorname{Im}\Psi$ is finite dimensional. This will prove that V_λ is finite dimensional.

Let ϕ be an eigenfunction for the eigenvalue λ and let ϕ_m be the corresponding radial eigenfunction around m. Then ϕ_m satisfies the differential equation (1.1). Thus $\phi_m \in V_\lambda$. Let $V_\lambda^+ = \{\sum_m a_m \phi_m : a_m \geqslant 0, m \in M\}$ where only finite sums are taken. V_λ^+ is the cone generated by $\{\phi_m, m \in M\}$.

Claim 1. $V_\lambda \cap C^\infty(B_m)$ is finite dimensional, i.e. $\operatorname{Im}\Psi$ is finite dimensional.

Proof. Let $h = \sum_1^k a_{m_i} \phi_{m_i} \in V_\lambda^+$. By the hypothesis of the theorem,

$$\phi_{m_i} = \cosh d(m_i, .) + D.$$

Thus

$$\nabla h = \sum_1^k a_{m_i} \sinh d(m_i, .) \frac{\partial}{\partial r_i},$$

where r_i is the distance function from the point m_i. Hence

$$|\nabla h|^2 = \sum_{i=1}^k a_{m_i}^2 (\sinh d(m_i, .))^2$$

$$+ 2 \sum_{i=1}^k a_{m_i} a_{m_j} \sinh d(m_i, .) \sinh d(m_j, .) \left\langle \frac{\partial}{\partial r_i}, \frac{\partial}{\partial r_j} \right\rangle.$$

But by Cauchy–Schwartz inequality $\langle \partial/\partial r_i, \partial/\partial r_j \rangle \leqslant 1$. Moreover $D > -1$ and $a_{m_i} \geqslant 0$. Hence we get

$$\begin{aligned} &|\nabla h|^2 \\ &\leqslant \sum_{i=1}^k a_{m_i}^2 \sinh^2 d(m_i, .) \\ &+ 2 \sum_{i=1}^k a_{m_i} a_{m_j} \sinh d(m_i, .) \sinh d(m_j, .) \\ &= \left(\sum_{i=1}^k a_{m_i} \sinh d(m_i, .) \right)^2 \\ &\leqslant C \left(\sum_{i=1}^k a_{m_i} \cosh d(m_i, .) + D \right)^2 \quad \text{where } C \text{ is some constant} \\ &= C|h|^2. \end{aligned}$$

Now we consider the cone $X = C^\infty(B_m) \cap V_\lambda^+$. Since $|\nabla h|_{B_m}| \leqslant |\nabla h|$, the above inequality shows that for $h \in X$, $|\nabla h| \leqslant C|h|$ holds. This shows the following: If W is an open ball in V_λ^+ such that $W \cap C^\infty(B_m)$ is a bounded family then it is a pointwise bounded equicontinuous family in X. Hence by the Ascoli–Arzela theorem it is relatively compact in X.

To prove the claim we now argue as follows: let U be the open ball around the origin in V_λ. Choose a g in V_λ such that $U \subset V_\lambda^+ - g$. The above argument shows that $(U+g) \cap C^\infty(B_m)$ is relatively compact in $V_\lambda^+ \cap C^\infty(B_m)$. Hence U is relatively compact in V_λ. Thus $V_\lambda \cap C^\infty(B_m)$ is finite dimensional and Claim 1 is proved.

Let $S = \{\phi_m : m \in M\}$ then $V_\lambda = \text{Spans } S$. Define

$$\tilde{B}: \mathrm{S} \times V_\lambda \to \mathbb{R}$$

by

$$\tilde{B}(\phi_m, \textstyle\sum a_n \phi_n) = \sum a_n \phi_n(m).$$

Then by the above observation there exists a unique $B: V_\lambda \times V_\lambda \to \mathbb{R}$, which extends $\tilde{B}$.

Claim 2. B is symmetric and nondegenerate.

Proof. Clearly B is symmetric. Nondegeneracy of B can be established as follows. Let $h \in \operatorname{Ker} B$ then

$$B(h, \phi_n) = 0, \text{ for all } n \in M.$$

Hence

$$h(n) = 0 \text{ for all } n \in M,$$

i.e., $h \equiv 0$ on M and hence B is nondegenerate. This completes the proof of Claim 2.

Now define the map

$$\Phi: M \to (V_\lambda, -B)$$

by

$$\Phi(m) := \phi_m = \cosh d(m, .) + D.$$

Let $\gamma(t)$ be a geodesic in M then

$$B(\phi_{\gamma(t)}, \phi_{\gamma(t)}) = \phi_{\gamma(t)}(\gamma(t)) = 1 + D.$$

This shows that $\Phi(M)$ is contained in $S = \{B(h, h) = 1 + D\}$, a 'sphere' in (V_λ, B). Further

$$B(\phi_{\gamma(t)}, \phi_{\gamma(t)}) = 1 + D$$

gives on twice differentiating,

$$B(\phi'_{\gamma(t)}, \phi'_{\gamma(t)})|_{t=0} + B(\phi''_{\gamma(t)}, \phi_{\gamma(t)})|_{t=0} = 0, \tag{2.1}$$

where $\phi'_{\gamma(t)} = \mathrm{d}/\mathrm{d}t(\phi_{\gamma(t)})$ etc. But

$$\phi_{\gamma(t)} = \cosh d(\gamma(t), .) + D.$$

Hence

$$B(\phi''_{\gamma(t)}|_{t=0}, \phi_{\gamma(0)}) = 1.$$

Therefore eq. (2.1) gives

$$B(\phi'_{\gamma(t)}, \phi'_{\gamma(t)})|_{t=0} = -1.$$

This shows that Φ is an isometric immersion of M into $(V_\lambda, -B)$. Finally, since $\cosh r + D$ is a monotone function, $\phi_m \neq \phi_n$ for $m \neq n$. Thus Φ is an imbedding of M into $(V_\lambda, -B)$.

Claim 3. $\Phi(M)$ is minimal in S.

Proof. Note that the Levi–Civita connection on $(V_\lambda, -B)$ is the flat Euclidean connection. Moreover the restriction of B to $T_p(\Phi(M))$, the tangent space of $\Phi(M)$, is positive definite. Thus $T_p(\Phi(M))^\perp$ is complementary to $T_p(\Phi(M))$. Now any vector can be uniquely decomposed into tangential and normal components so that the second fundamental form can be defined. Hence it makes sense to define the mean curvature of $\Phi(M)$ as the trace of the second fundamental form. Now since the imbedding is given by eigenfunctions we conclude that $\Phi(M)$ is minimal in S by a result of [6], pp. 340.

We now compute the densities of these harmonic manifolds. Recall that the function $f(r) = \cosh r + D$ satisfies the equation

$$f'' + \frac{\theta'}{\theta} f' + \lambda f = 0.$$

Hence we get

$$\frac{\theta'}{\theta}(r) \sinh r = -(1+\lambda)\cosh r - \lambda D$$

which gives

$$\theta(r) = 2^{-(1+\lambda)} \left(\sinh\left(\frac{r}{2}\right)\right)^{-(1+\lambda+\lambda D)} \left(\cosh\left(\frac{r}{2}\right)\right)^{-(1+\lambda-\lambda D)}.$$

But as $r \to 0$, $\theta(r) \to (\sinh(r/2))^{d-1}$, hence we get

$$1 + \lambda + \lambda D = 1 - d, \quad \text{and} \quad \theta(r) = 2^{(d-1)} \left(\sinh\left(\frac{r}{2}\right)\right)^{d-1} \left(\cosh\left(\frac{r}{2}\right)\right)^{(d-1+2\lambda D)}.$$

Let us now use the fact that harmonic manifolds are Einstein. Let Ricci $(M) = k$. Now

$$\theta(r) = r^{d-1}\omega(r)$$

gives

$$\frac{\theta'(r)}{\theta(r)} = \frac{d-1}{r} + \frac{\omega'(r)}{\omega(r)}.$$

Using the expression for θ obtained above, we get

$$\frac{\omega'(r)}{\omega(r)} = -(1+\lambda)\coth r - \frac{\lambda D}{\sinh r} - \frac{n-1}{r}.$$

Differentiating twice and after simplification, we get

$$-\frac{1}{3}\operatorname{Ricci}(M) = \frac{d-1}{3} + \frac{\lambda D}{2}$$

which gives

$$\lambda D = -\frac{2}{3}(k + d - 1), \quad \text{but } 1 + \lambda + \lambda D = 1 - d$$

hence

$$\lambda = -\frac{2}{3}\left(1 - k + \frac{d}{2}\right) \quad \text{and} \quad D = \frac{k+d-1}{1-k+(d/2)}.$$

Finally the above computations show that

$$\begin{aligned} f(r) &= \cosh r + D \\ &= \cosh r - 1 + \frac{3d/2}{1-k+(d/2)} \\ &= 2\sinh^2\left(\frac{r}{2}\right) + \frac{3d/2}{1-k+(d/2)} \\ &\geqslant \frac{3d/2}{1-k+(d/2)}. \end{aligned}$$

Thus f has no zeroes on M. This completes the proof of the theorem. ■

3. Symmetry of M

Let M satisfy the hypothesis of Theorem 1.1. We prove the symmetry of M under some assumptions. Two cases arise.

Case 1. $D = 0$

In this case M is symmetric, since the Sturm–Liouville equation

$$f'' + \frac{\theta'}{\theta} f' + \lambda f = 0$$

shows that

$$\theta(r) = \sinh^{d-1} r$$

and Theorem (4.2.9) shows that M is isometric to the real hyperbolic space with constant curvature -1.

Case 2. $D \neq 0$

Let $\gamma(t)$ be geodesic of M and ϕ be an eigenfunction for the eigenvalue λ. Let V_γ be the subspace of V_λ generated by $\{\phi_{\gamma(t)} : t \in \mathbb{R}\}$.

Lemma 3.1. *Let M satisfy the hypothesis of Theorem* (1.1). *Assume in addition that B restricted to the subspace V_γ is nondegenerate. Then M is symmetric.*

Proof. Fix an imbedding $\Phi : M \to V_\lambda$; $\Phi(m) = \phi_m$. Let $\gamma(t)$ be a geodesic on M. Then

$$B(\phi_{\gamma(t)}, \phi_{\gamma(0)}) = \cosh t + D.$$

Differentiating we get

$$B(\phi'''_{\gamma(t)} - \phi'_{\gamma(t)}, \phi_{\gamma(0)}) = 0.$$

Now $\gamma(0)$ is an arbitrary point of M and $\phi'''_{\gamma(t)} - \phi'_{\gamma(t)} \in V_\gamma$. Since by our assumption nondegenerate on V_γ, the geodesic γ satisfies

$$\phi'''_{\gamma(t)} - \phi'_{\gamma(t)} = 0.$$

Take $m \in M$. Note that $B|_{T_m(M)}$ is positive, hence we can take the orthogonal comp ment N_m with respect to B, of $T_m(M)$ to M. Define

$$\tau_m : V_\lambda \to V_\lambda$$

to be the reflection with respect to the subspace N_m. τ_m is an isometry of V_λ, it fixes m reverses the geodesics through m. Moreover it also leaves M invariant. Hence it indu an isometry on M which is obviously the geodesic involution of M. Thus M a symmetric space.

The next lemma shows that V_γ is 3-dimensional if and only if B is nondegenerate o

Lemma 3.2. *Let M satisfy the hypothesis of our theorem. Let γ be a geodesic, then* 3-*dimensional if and only if B is nondegenerate on V_γ.*

Proof. Fix an imbedding $\Phi: M \to V_\lambda: \Phi(m) = \phi_m$ where ϕ is an eigenfunction for eigenvalue λ. Let $s_i = -1, 0, 1$ be such that $\{\phi_{\gamma(s_i)}\}_1^3$ generate V_γ. Let $g \in V_\gamma$ sat $B(g, \phi_{\gamma(t)}) = 0$ for all t. Put

$$g = \sum_1^3 a_i \phi_{\gamma(s_i)}$$

$$= \sum_1^3 a_i \cosh d(\gamma(s_i), .) + D \sum_1^3 a_i.$$

Now $B(g, \phi_{\gamma(t)}) = 0$ gives $g(\gamma(t)) = 0$ for all t, i.e

$$\sum a_i D + \sum a_i \cosh(s_i - t) = 0, \text{ for all } t.$$

Dividing by $\cosh t$ and letting $t \to \infty$ we see that

$$\sum a_i = 0,$$

hence

$$\sum a_i \cosh(s_i - t) = 0, \text{ for all } t.$$

Expanding and equating the coefficients of $\cosh t$ and $\sinh t$ to 0 we get

$$\sum a_i \cosh s_i = 0 = \sum a_i \sinh s_i.$$

Substituting the values of s_i and using the equation $\sum a_i = 0$ we obtain a system of th equations

$$a_0 \cosh 1 + a_1 + a_2 \cosh 1 = 0,$$

$$-a_0 \sinh 1 + 0 + a_2 \sinh 1 = 0,$$

$$a_0 + a_1 + a_2 = 0.$$

But the coefficient matrix of the above system is nonsingular, hence $a_i = 0$ for all $g \equiv 0$ or B is nondegenerate.

Conversely let B be nondegenerate on V_λ. As in proof of lemma (5.3.6), we see that $\phi_{\gamma(t)}$ satisfies the differential equation

$$\phi'''_{\gamma(t)} - \phi'_{\gamma(t)} = 0.$$

Thus V_γ is 3 dimensional. Combining the above two lemmas one sees that M is symmetric if V_γ is 3-dimensional.

4. Remarks

For manifolds which satisfy the hypothesis of the theorem the density $\theta(r)$ is given by

$$\theta(r) = 2^{d-1}\left(\sinh\frac{r}{2}\right)^{d-1}\left(\cosh\frac{r}{2}\right)^{d-1-(4/3)(k+d-1)}.$$

Put $b = d - 1 - (4/3)(k + d - 1)$. By the Bishop–Gromov comparison theorem ([5] or [7]), b satisfies $0 \leqslant b \leqslant d - 1$.

1. If $b = d - 1$, $\theta(r) = \sinh^{d-1} r$ and hence M is isometric to the real hyperbolic space. In this case $D = 0$.
2. If $b = 0$, $\theta(r) = 2^{d-1}(\sinh(r/2))^{d-1}$. Again M is isometric to the real hyperbolic space. But in this case $D \neq 0$ and the eigenfunction $f_p = \cosh r + D$ becomes the eigenfunction for the next eigenvalue.
3. All the other harmonic spaces which satisfy the hypothesis of the theorem, in particular the NA spaces have $0 < b < d - 1$.

Question. Is b an integer? It is for the NA spaces.

Acknowledgement

The author K Ramachandran was supported by the National Board for Higher Mathematics, DAE, India.

References

[1] Besse A L, *Manifolds all of whose geodesics are closed*, Berlin: Springer (1978)
[2] Besse A L, *Einstein Manifolds*, (Berlin: Springer) (1987)
[3] Damek E and Ricci F, A class of nonsymmetric harmonic Riemmanian spaces, *Bull. AMS* **27** (1992) 139–142
[4] Damek E and Ricci F, Harmonic analysis on solvable extension of H-type groups, *J. Geom. Anal.* **2** (1992) 213–247
[5] Gallot S, Hulin D and Lafontaine J, *Riemannian Geometry* (Berlin: Springer) (1990)
[6] Kobayashi S and Nomizu K, *Foundations of Differential Geometry, II* (New York: Wiley Inter-science) (1969) p. 340
[7] Kumaresan S, *A Course in Riemannian Geometry*, Lecture Notes, Instructional Workshop on Riemannian Geometry, T.I.F.R., Bombay 1990
[8] Ramachandran K and Ranjan A, *Curvature and helical immersions of harmonic manifolds*, Preprint, 1995
[9] Ramachandran K and Ranjan A, Harmonic manifolds with some specific volume densities, *Proc. Indian Acad. Sci.* **107** (1997) 251–261
[10] Szabo Z I, The Lichnerowicz Conjecture on Harmonic Manifolds, *J. Diff. Geom.* **31** (1990) 1–28
[11] Szabo Z I, Spectral theory for operator families on Riemannian manifolds, *Proc. Symp. Pure Math.* **54** (1993) 615–665

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 1, February 1998, pp. 23–30.

Remarks on Banaschewski–Fomin–Shanin extensions

VISHVAJIT V S GAUTAM and VINOD KUMAR

Department of Mathematics, Kurukshetra University, Kurukshetra 136 119, India
E-mail: kuru@doe.ernet.in

MS received 10 April 1997; revised 9 October 1997

Abstract. The notion of B^*-continuous and B_c^*-continuous maps is introduced. The problem of epireflection of Banaschewski–Fomin–Shanin extension for a general Hausdorff space is investigated with the help of ${}_pB^*$ and ${}_pB_c^*$-continuous maps.

Keywords. BFS-extension; H-closed extension; minimal Hausdorff extension.

1. Introduction

The concept of H-closed spaces is introduced by Alexandroff and Urysohn [1]. On the line of Stone–Cech compactification the reflective properties of H-closed spaces are studied by many researchers ([6, 8, 9, 10, 13]). Herrlich and Strecker [6] have shown that Katětov extension k is not a functor from the category of Hausdorff spaces and continuous maps to the full subcategory of H-closed spaces. Harris [5] showed that if we restrict the class of continuous maps to p-maps then k is the required reflective functor from the category of Hausdorff spaces and p-maps to the full subcategory of H–closed spaces. Between 1960 to 1980, a good amount of work was done on categorical aspects to H-closed extensions. But, in case of minimal Hausdorff extensions, work carried out by researchers is of local nature, i.e., certain properties had to hold at each point. Problems related with categorical aspects of minimal Hausdorff spaces remain largely open. Banaschewski [2] proved that a Hausdorff space has a minimal Hausdorff extension if and only if it is semiregular. Each semiregular Hausdorff space X has largest minimal Hausdorff extension (cf. [14, 15]) denoted by μX and called the Banaschewski–Fomin–Shanin Hausdorff extension of X. We do not have a suitable answer regarding epireflection of the category of semiregular Hausdorff spaces to the full subcategory of minimal Hausdorff spaces, μ as a reflection. In this paper we give an answer to this problem with the help of some new class of maps.

2. Preliminaries

Throughout this paper all spaces assumed are to be Hausdorff. If A is a subset of a space X, then $\mathrm{int}_X(A)$ (resp. $\mathrm{cl}_X(A)$) will denote the interior (resp. closure) of A in X. A space X is called H-closed if every homeomorphic image of X which is a subset of a Hausdorff space Y, is a closed subset of Y.

For a Hausdorff space, X the semiregularization of X, denoted by X_s, is the space generated by the open basis $\{\mathrm{int}(\mathrm{cl}\, U) \mid U \text{ open in } X\}$, X is semiregular iff $X = X_s$.

A space Y is an extension of X if X is a dense subspace of Y; if Y possesses some topological property P, then Y is a P-extension of X. Let Y be a Hausdorff extension of X. For $y \in Y$, O^y (sometimes written as O_Y^y) denotes $\{U \cap X \mid y \in U,\ U \text{ open in } Y\}$;

$\{O^y | y \in Y\}$ is called the neighborhood filter trace of Y on X. Then O^y is an open filte
X, and for $x \in X$, O^x is precisely the open neighborhood system η_x of x. For an o
subset U of X, let oU denote $\{y \in Y | U \in O^y\}$. The set $\{U \cup \{y\} | U \in O^y, y \in Y\}$ and $\{oU$
open in $X\}$ form a bases for topologies on Y, and the resulting new spaces are deno
by Y^+ and $Y^\#$, respectively. Now Y^+ and $Y^\#$ are also Hausdorff extensions of X,
the topology of $Y^\#$ (resp. Y) is coarser than the topology of Y (resp. Y^+) [3]; in fact,
and $Y^\#$ are H-closed if and only if Y is H-closed. Y is called *simple* (resp. *strict*) *extens*
if $Y = Y^+$ (resp. $Y = Y^\#$).

Let $X^* = \{\mathscr{U} | \mathscr{U}$ is a free open ultra filter on $X\}$. For each open subset U of X
$O_U = U \cup \{\mathscr{U} \in X^* - X | U \in \mathscr{U}\}$ X^* with the topology generated by the open b
$\{O_U | U$ open in $X\}$ is an H-closed extension [4] of X denoted by σX and called *Fo*
extension of X; X^* with the topology generated with the open basis $\{U | U$ ope
$X\} \cup \{U \cup \{\mathscr{U}\} | U$ open in X, $U \in \mathscr{U}$, $\mathscr{U} \in X^* - X\}$ is an H-closed extension [8
X denoted by kX and called *Katětov extension* of X.

If X is H-closed, then X_s is minimal Hausdorff and the topology of X_s is the smal
of the Hausdorff topology coarser than the topology of X. We recall that a space
said to be minimal Hausdorff if its topology does not contain any coarser Hausd
topology on X.

The following results are in [16].

1) For a Hausdorff space X the following are equivalent:

(i) X is minimal Hausdorff,
(ii) X is semiregular and H-closed, and
(iii) every open filter with a unique accumulation point converges.

2) (a) A space can be densely embedded in a minimal Hausdorff space if and only if
semiregular.
(b) Any space can be embedded as a closed nowhere dense subspace of a mini
Hausdorff space.

3. BFS-extension for a general Hausdorff space

If X is semiregular, then its minimal Hausdorff extension is denoted by μX and is ca
the Banaschewski–Fomin–Shanin extension (BFS-extension) of X. $(kX)_s$ and $(\sigma X)_s$
both minimal Hausdorff extension of X and we also have $(kX)_s = (\sigma X)_s$. BFS-extens
has been studied by many researchers for a semiregular space X ([11, 14, 15]). Ti
[17] constructed an extension of the type μX for a general Hausdorff space X.

Let X be a Hausdorff space and let

$$\hat{X} = X \cup \{\mathscr{U}_s | \mathscr{U} \text{ is a free open ultrafilter on } X\},$$

where $\mathscr{U}_s$ is generated by the filter base $\{\text{int}(\text{cl}\, U) | U \in \mathscr{U}\}$.

For each G open in X, let

$$\hat{o}(G) = G \cup \{\mathscr{U}_s | \mathscr{U}_s \in \hat{X} - X, \quad G \in \mathscr{U}_s\},$$

the family $\{\hat{o}(G) | G$ open in $X\}$ form a base for a topology $\mathscr{J}^\#$ on X. A rout
verification shows that $(\hat{X}, \mathscr{J}^\#)$ is a strict H-closed extension of X.

We define a topology $\mathscr{J}^+$ on $\hat{X}$ by declaring that X is open in $\hat{X}$, and for $\mathscr{U}_s \in \hat{X} -$
a $\mathscr{J}^+$-basic neighborhood of $\mathscr{U}_s$ is $U \cup \{\mathscr{U}\}$, where U is open in X and $U \in \mathscr{U}_s$. T
$(\hat{X}, \mathscr{J}^+)$ is a simple H-closed extension of X.

Note that, if $\alpha:\sigma X \to \hat{X}$ is defined by

$$\alpha(x) = x \quad \text{if } x \in X,$$
$$\alpha(\mathcal{U}) = \mathcal{U}_s \quad \text{if } \mathcal{U} \in \sigma X - X,$$

then α is a bijection.

To see the characterization of the extension $\hat{X}$ for a semiregular space X, we have the following [17].

3.1 PROPOSITION

The following statements are equivalent for a Hausdorff space X.

(i) *X is semiregular,*
(ii) *$\hat{X}$ is semiregular,*
(iii) $\hat{X} = (\sigma X)_s = (kX)_s$.

In view of the above proposition we denote $\hat{X}$ by μX, the BFS-extension of X.

The idea of θ-continuity is due to Fomin [4]. These maps have particular importance in our study on BFS-extension. We recall the following:

3.1 DEFINITION

A map $f: X \to Y$ is said to be θ-continuous if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(\mathrm{cl}_X U) \subset \mathrm{cl}_Y V$. Of course, continuity implies θ-continuity.

The converse is true if we restrict our space, e.g., if Y is regular, Note that the identity map $i: X_s \to Y$, where X_s is the semiregularization of a non semiregular space X, is an example of θ-continuous map which is not continuous.

Composition of two θ-continuous maps is θ-continuous. This fact helps us to construct categories with θ-continuous maps.

In categorical point of view, it is important to find out the condition when a θ-continuous map with semiregular codomain is continuous. Following proposition gives an answer to this.

3.3 PROPOSITION

Let $f: X \to Y$ be a θ-continuous map into a semiregular space. If $f(A) \subset \mathrm{int}_Y \, \mathrm{cl}_Y (f(\mathrm{cl}_X A))$ for all A open in X, then f is continuous.

Proof. The proof is straightforward and is omitted.

Another important map in connection with H-closed extension are p-maps. An intensive application of these maps can be found in Harris [5].

3.4 DEFINITION

A p-cover of a space is an open cover such that the union of some finite subcollection is dense, a p-map is a continuous map such that the inverse of a p-cover of the codomain is a p-cover of the domain.

Every map with H-closed domain is a p-map.

Now we are giving the definition of B^*-continuous and B_c^*-continuous maps. Necessary application of these maps can be seen in the next section. But in order to maintain continuity with subject matter we are defining these here.

Let X^* and Y^* be the extensions of X and Y respectively. Here $*$ stands for the extension under consideration (e.g. $*$ may be a Hausdorff extension [4], a Katětov extension, a Fomin extension, or a BFS-extension).

3.5 DEFINITION

A continuous function $f:X \to Y$ is called B^*-continuous if f has an extension (not necessarily continuous) $f^*:X^* \to Y^*$ such that $f^*(A) \subset \mathrm{int}_Y * \mathrm{cl}_Y * (f^*(\mathrm{cl}_X * A))$ for all A open in X^*. f is said to be B_c^*-continuous if f^* is also continuous.

The B_c^*-continuous maps are of real importance. In order to characterize these maps we have the following.

3.6 PROPOSITION

If $f:X \to Y$ is B_c^*-continuous, then $f(A \cap X) \subset \mathrm{int}_Y\, \mathrm{cl}_Y\, (f(A \cap X))$ for all A open in X^*.

Proof. Since $f:X \to Y$ is B_c^*-continuous, there exists a unique continuous extension $f^*:X^* \to Y^*$ of f such that $f^*(A) \subset \mathrm{int}_Y * \mathrm{cl}_Y * (f^*(\mathrm{cl}_X * A))$ for all A open in X^*.

We have $f^*(A) \subset \mathrm{int}_Y * \mathrm{cl}_Y * (f^*(\mathrm{cl}_X * A))$, for all A open in X^*. This gives

$$f^*(A \cap X) \subset \mathrm{int}_Y * \mathrm{cl}_Y * (f^*(\mathrm{cl}_X * (A \cap X))$$

or

$$\begin{aligned} f(A \cap X) \cap Y \subset\ & \mathrm{int}_Y * (\mathrm{cl}_Y * (f^*(\mathrm{cl}_X * (A \cap X))) \cap Y \\ &= \mathrm{int}_Y(\mathrm{int}_Y * \mathrm{cl}_Y * (f^*\mathrm{cl}_X * (A \cap X))) \cap Y) \\ &\subset \mathrm{int}_Y(\mathrm{cl}_Y * (f^*\mathrm{cl}_X * (A \cap X))) \cap Y) \\ &\subset \mathrm{int}_Y\, \mathrm{cl}_Y(\mathrm{cl}_Y * (f^*\mathrm{cl}_X * (A \cap X))) \cap Y) \\ &= \mathrm{int}_Y\, \mathrm{cl}_Y(f(A \cap X)). \end{aligned}$$

To see the converse, we have

3.7 PROPOSITION

Let $f:X \to Y$ be continuous and let $f^*:X^* \to Y^*$ be the unique continuous extension of f. If $\mathrm{cl}_Y * f(A \cap X) \subset \mathrm{int}_Y * \mathrm{cl}_Y * (f(\mathrm{cl}_X(A \cap X))$ for all A open in X^*. Then $f^*(A) \subset \mathrm{int}_Y * \mathrm{cl}_Y * (f^*(\mathrm{cl}_X * A))$ i.e. f is B_c^*-continuous.

Proof. We have $\mathrm{cl}_Y * f(A \cap X) \subset \mathrm{int}_Y * \mathrm{cl}_Y * (f(\mathrm{cl}_X(A \cap X))$. This gives

$$\mathrm{cl}_Y * f^*(A \cap X) \subset \mathrm{int}_Y * \mathrm{cl}_Y * (f^*(\mathrm{cl}_X * (A \cap X)),$$

$$\mathrm{cl}_Y * f^*(A \cap X) \subset \mathrm{int}_Y * \mathrm{cl}_Y * (f^*(\mathrm{cl}_X * A),$$

$$f^*(\mathrm{cl}_X * (A \cap X)) \subset \mathrm{int}_Y * \mathrm{cl}_Y * (f^*(\mathrm{cl}_X * A)),$$

$$f^*(\mathrm{cl}_X * A) \subset \mathrm{int}_Y * \mathrm{cl}_Y * (f^*(\mathrm{cl}_X * A),$$

$$f^*(A) \subset \mathrm{int}_Y * \mathrm{cl}_Y * (f^*(\mathrm{cl}_X * A)).$$

A p-map which is also B^*-continuous (B_c^*-continuous) said to be pB^*-continuous (pB_c^*-continuous).

Following proposition is due to Banaschewski [2], which shows that to get a continuous extension from μX to minimal Hausdorff extension Y of X, we have to restrict our continuous maps from X to Y.

3.8 PROPOSITION

For each minimal Hausdorff extension Y of a space X there exists a θ-continuous map completing the following diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr] & Y \\ & \mu X \arrow[u] \end{tikzcd} \qquad (*)$$

An example given in [2], shows that $\mu X \to Y$ in $(*)$ in general, is not continuous.
We strengthen Banaschewski's result in the following proposition.

3.9 PROPOSITION

If $f: X \to Y$ is a p-map into a H-closed space, then there exists a θ-continuous map completing the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \\ & \mu X \arrow[u] \end{tikzcd}$$

Proof. Let $f: X \to Y$ be a p-map and let Y be H-closed. Since for each H-closed extension Y of X there is a unique continuous function from kX onto Y that leaves X pointwise fixed, there exists a unique map $g: kX \to Y$ such that $g \circ k = f$. The identity map $i: \sigma X \to kX$ is θ-continuous, therefore $g \circ i: \sigma X \to Y$ is θ-continuous. Define $j: \mu X \to \sigma X$ as follows:

$$\begin{aligned} j(x) &= x && \text{if} \quad x \in X \\ j(\mathscr{U}_s) &= \mathscr{U} && \text{if and only if } \alpha(\mathscr{U}) = \mathscr{U}_s \text{ for all } \mathscr{U}_s \in \hat{X} - X. \end{aligned}$$

By a routine verification we can prove that j is a θ-continuous map. Since $g \circ i: \sigma X \to Y$ and $j: \mu X \to \sigma X$ are θ-continuous maps therefore the composite map $j \circ g \circ i$ is the required θ-continuous map.

One of the basic problem related to extension properties of topological spaces is that: a continuous map between two spaces can be extended to a map between their given extension spaces, such as their Katětov extensions. In the case of BFS-extension we have the following theorem (cf. [7] and use 3.3).

3.10 Theorem. *Let $f: X \to Y$ be a p-map and let f^* be a function on μX to μY defined as follows*

$$f^*(x) = x \qquad \text{if } x \in X$$

$$f^*(\mathscr{U}_s) = \begin{cases} \mathscr{N}_{f(x)} & \text{if } \mathscr{U}_s = \mathscr{N}_x \\ \mathscr{N}_{y_0} & \text{if } f^0(\mathscr{U}_s) \longrightarrow y_0, \\ f^0(\mathscr{U}_s) & \text{otherwise} \end{cases}$$

$$\text{if } \mathscr{U}_s \in \mu X - X,$$

where $f^0(\mathscr{U}_s) = \{V \in \{S \subset Y \mid \operatorname{int}\operatorname{cl}(S) = S\} \mid f^{-1}(V) \in \mathscr{U}_s\}$ and $\mathscr{N}_x$ be the open neighborhood system of x.

In addition, if f^* satisfies the condition $f^*(A) \subset \operatorname{int}_Y * \operatorname{cl}_Y * (f^*(\operatorname{cl}_X * A)$ for all A open in X^*. Then $f^*: \mu X \to \mu Y$ is a unique continuous extension of f.

Theorem 3.10 can be restated as follows:

3.11 Theorem. *If X and Y are semiregular and $f: X \to Y$ is pB^*-continuous, then there exists a unique extension $f^*: \mu X \to \mu Y$ of f which is also a pB_c^*-continuous.*

4. Categorical aspects of BFS

A functor $\mathbf{r}$ from a category $\mathscr{C}$ to a subcategory $\mathscr{D}$ of $\mathscr{C}$ is a reflective functor if there is a morphism $f_c: C \to \mathbf{r}C$ and every morphism g from C to an object D of $\mathscr{D}$ factors uniquely through $\mathbf{r}C$ via f_c so that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "f_C"] \arrow[dr, "g"'] & rC \arrow[d, "h!"] \\ & D \end{tikzcd}$$

$$h \circ f_C = g.$$

If $\mathbf{r}: \mathscr{C} \to \mathscr{D}$ is a reflective functor, the subcategory $\mathscr{D}$ is called a reflective subcategory. The object $\mathbf{r}C$ of $\mathscr{D}$ is called the reflection of $\mathscr{C}$ in $\mathscr{D}$. If $\mathbf{r}$ is an epimorphism, the subcategory $\mathscr{D}$ is called the epireflective subcategory of the category $\mathscr{C}$.

The Stone–Cech compactification is an example of epireflective functor from the category of completely regular Hausdorff spaces to the category of compact Hausdorff spaces.

The study of epireflective subcategories of Hausdorff spaces is an important part of topological research. One of the well known result related to H-closed extension is due to Harris [5]. He proved that the category of H-closed spaces and maps is an epireflective subcategory of the category of Hausdorff spaces and p-maps, and the epireflection of a space is its Katětov estension. In the case of BFS-extension Hunsaker and Naimpally [7] have shown that the category of minimal Hausdorff spaces and continuous maps is an epireflective subcategory of the category of semiregular spaces and λ-perfect maps. But by theorem 4.4 of [7], composition of two λ-perfect maps is not λ-perfect map unless the λ-perfect map takes free open ultrafilters to free open ultrafilters. Therefore theorem 4.5 of [7] is required additional restriction on maps to get the respective categories. Also the way in which theorem 4.5 of [7] has been stated shows that maps of the subcategories do not belong to their respective categories.

In order to get a suitable answer we start with the following:

.1 *Lemma. If $\mathscr{M}_0$ denotes the collection of minimal Hausdorff spaces and p-maps aving the property; for every f in $\mathscr{M}_0$, $f(A) \subset \operatorname{int}\operatorname{cl} f(\operatorname{cl} A)$ for all A open in domain f. 'hen $\mathscr{M}_0$ forms a category.*

'roof. It is enough to show that composition of two maps in $\mathscr{M}_0$ is again a map in $\mathscr{M}_0$. et f and g be two maps in $\mathscr{M}_0$ with codomain $f =$ domain g, and let A be open in the omain f.

$$\begin{aligned} gf(A) = g(f(A)) &\subset g(\operatorname{int}(\operatorname{cl}(f(\operatorname{cl} A)))) \\ &\subset \operatorname{int}\operatorname{cl}(g(\operatorname{cl}\operatorname{int}\operatorname{cl}(f(\operatorname{cl} A)))) \\ &\subset \operatorname{int}\operatorname{cl}(g(\operatorname{cl}(f(\operatorname{cl} A)))) \\ &\subset \operatorname{int}\operatorname{cl}(gf(\operatorname{cl} A)). \end{aligned}$$

'his shows that gf is a map in $\mathscr{M}_0$.

.2 *Lemma. If $\mathbb{S}$ denotes the collection of semiregular spaces and pB_c^*-continuous maps ıen $\mathbb{S}$ forms a category.*

'roof. Use of the fact that composition of two pB_c^*-continuous maps is again B_c^*-continuous.

.3 *Lemma. The category $\mathscr{M}$ of minimal Hausdorff spaces and pB_c^*-continuous maps is full subcategory of $\mathbb{S}$.*

By theorem 3.11 and the above lemmas we have the following.

.4 Theorem. *$\mathscr{M}$ is an epireflective subcategory of $\mathbb{S}$. Moreover, $\mathscr{M}$ is the largest ıbcategory of $\mathbb{S}$ in the sense of Porter* [12].

Iote that, a monomorphism j is an extremal monomorphism if whenever we have the ommutative diagram as illustrated so that e is an epimorphism, then e is an omorphism. The object X is said to be extremal subobject of Y.

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "e"'] & Y \\ & Z \arrow[u] \end{tikzcd}$$

Since the extremal subobjects in the category of Hausdorff spaces are the closed ıbspaces, and the epireflective subcategories are closed under extremal subobjects. 'herefore it is interesting to see the extremal subobjects in the epireflective subategories. In this connection we have the following.

.5 *Problem.* Characterize the extremal subobjects of $\mathscr{M}$ in $\mathbb{S}$.

Since the result stated in theorem 3.10 is also true for strict extension of Fomin type, ıerefore theorem 3.11 can be restated as, if $f: X \to Y$ is pB_c^*-continuous, then there

exists a unique extension $f^*\colon \sigma X \to \sigma Y$ of f which is also a pB_c^*-continuous. On the line of theorem 4.4 we have the following theorem.

4.6 Theorem. *The category of H-closed spaces and pB_c^*-maps is an epireflective subcategory of Hausdorff spaces and pB_c^*-maps, σ as an epireflection.*

Acknowledgement

The authors are thankful to the referee for his valuable and very helpful suggestions. Research of the author (VVSG) was supported by University Grants Commission.

References

[1] Alexandroff P and Uryohn P, Mémoire sur les Espaces Topologiques Compacts, *Verh. Nederl. Akad. Wetensch. Afd. Natuurk.* **14** (1929) 1–96

[2] Banaschewski B, Über Hausdorff sch-minimale Erweite-rrungen von Räumen, *Arch. Math.* **12** (1961) 355–365

[3] Banaschewski B, Extension of Topological Spaces, *Canad. Math. Bull.* **7** (1964) 1–22

[4] Fomin S, Extension of topological spaces, *Ann. Math.* **44** (1943) 471–480

[5] Harris D, Katětov extension as a functor, *Math. Ann.* **193** (1971) 171–175

[6] Harrlich H and Strecker G E, *H*-closed spaces and reflective subcategories, *Math. Ann.* **177** (1968) 302–309

[7] Hunsaker W N and Naimpally S A, Extension of continuous functions: Reflective functors, *Bull. Austral. Math. Soc.* **35** (1987) 455–470

[8] Katětov M, Über *H*-abgischlossene and bikompate raüme, *Časopis Pěst. Mat.* **69** (1940) 36–49

[9] Liu C T, Absolutely closed spaces, *Trans. Am. Math. Soc.* **130** (1968) 86–104

[10] Mioduszewski J and Rudolf L, *H*-closed and extremally disconnected Hausdorff spaces, *Disseertation Mathematicae, Rozprawy Mat.* **66** (1969) 1–52

[11] Porter J R, On locally *H*-closed spaces, *Proc. London Math. Soc.* **20** (1970) 193–204

[12] Porter J R, Extension function on subcategories of HAUS, *Canad. Math. Bull.* **18(4)** (1975) 587–590

[13] Porter J R and Thomas J, On *H*-closed and minimal Hausdorff spaces, *Trans. Am. Math. Soc.* **138** (1969) 159–170

[14] Porter J R and Votaw C, *H*-closed extension I, *Gen. Top. Appl.* **3** (1973) 211–224

[15] Porter J R and Votaw C, *H*-closed extension II, *Trans. Am. Math. Soc.* **202** (1975) 193–209

[16] Porter J R and Wood G, *Extensions and Absolutes of Hausdorff spaces* (Springer–Verlag) (1988)

[17] Tikoo M L, The Banaschewski–Fomin–Shanin extension μX, *Topology Proc.* **10** (1985) 187–206

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 1, February 1998, pp. 31–40.

Weyl multipliers for invariant Sobolev spaces

RAMAKRISHNAN RADHA and SUNDARAM THANGAVELU*

Department of Mathematics, Indian Institute of Technology, Madras 600 036, India
*Stat-Math. Unit, Indian Statistical Institute, 8th Mile, Mysore Road, Bangalore 560 059, India
E-mail: radharam@imsc.ernet.in; veluma@isibang.ernet.in

MS received 3 February 1997; revised 1 December 1997

Abstract. A concrete characterization for the L^p-multipliers $(1 < p < \infty)$ for the Weyl transform is obtained. This is used to study the Weyl multipliers for Laguerre Sobolev spaces $W^{m,p}(\mathbb{C}^n)$. A dual space characterization is obtained for the Weyl multiplier class $M_W(W_L^{m,1}(\mathbb{C}^n))$.

Keywords. Heisenberg group; Hilbert–Schmidt operator; multiplier; Sobolev space; special Hermite functions; twisted convolution; Weyl transform.

1. Introduction

The Laguerre Sobolev spaces $W_L^{s,2}(\mathbb{C}^n)$ were introduced by Peetre and Sparr in [6]. They were also studied by Thangavelu [9] in connection with the spherical means of the Heisenberg group. This space has an invariant property which is not shared by the usual Sobolev space $W^{s,2}(\mathbb{R}^n)$ namely, it is invariant under the symplectic Fourier transform. The details and the relation between this space and the usual Sobolev space can be found in [10].

Fourier multipliers for ordinary Sobolev spaces $W^{m,p}(\mathbb{R}^n)$, $(m \geqslant 0$, an integer), $1 \leqslant p < \infty$ have been characterized by Poornima in [7]. The purpose of this paper is to consider a similar problem for Weyl multipliers for the Laguerre Sobolev spaces $W_L^{m,p}(\mathbb{C}^n)$.

This paper is organized in the following way: In §2, we give the required notations and collect the necessary background. In §3, we obtain a concrete characterization for the L^p-multipliers $(1 < p < \infty)$ for the Weyl transform. In §4, we characterize the Weyl multipliers for $W_L^{m,p}(\mathbb{C}^n)$, based on the result which we obtain in §3. In §5, a dual space characterization is obtained for the space $M_W(W^{m,1}(\mathbb{C}^n))$.

2. Notations and preliminaries

Characterization of Fourier multipliers of L^p-spaces is one of the important problems in multiplier theory. For definition, examples and sufficient conditions for L^p-multipliers on $\mathbb{R}^n$, we refer to Stein [8]. A necessary condition, namely if m is a multiplier for $L^p(\mathbb{R}^n)$, then there exists a pseudo measure σ such that $T_m f = \sigma * f$(* denotes convolution) is also known. In fact, this result is proved for any locally compact abelian group G in place of $\mathbb{R}^n$. This is based on the development of the works of Hormander [3] and Gaudry [2]. The details can be found in [4].

The Weyl transform $W(f)$ of a function $f \in L^1(\mathbb{C}^n)$ is defined by

$$W(f)\varphi(\xi) = \int_{C^n} f(z)\exp(ix(y/2 + \xi))\varphi(\xi + y)\,\mathrm{d}z, \quad \varphi \in L^2(\mathbb{R}^n)$$

where $z = x + iy$. The map W from $L^1(\mathbb{C}^n)$ to the space of bounded operators
$L^2(\mathbb{R}^n)$, defined as above, extends uniquely to a bijection from $S'(\mathbb{C}^n)$ to the space
continuous linear maps from $S(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$. Moreover, W maps $L^2(\mathbb{C}^n)$ unita
onto the space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$. In other words, we have
Plancherel formula for the Weyl transform, given by

$$\|f\|_2^2 = (2\pi)^{-n} \|W(f)\|_{\mathrm{HS}}^2.$$

The inversion formula is given by

$$f(z) = (2\pi)^{-n} \mathrm{tr}(W(z)^* W(f)),$$

where $W(z)$ is the operator valued function

$$W(z)\varphi(\xi) = \exp(ix(y/2 + \xi))\varphi(\xi + y).$$

For details, we refer to Folland [1].

The twisted convolution of two functions $f, g \in L^1(\mathbb{C}^n)$ is defined by

$$f \times g(z) = \int_{C^n} f(z - w) g(w) \exp(i \mathrm{Im}\, z\bar{w}/2) \mathrm{d}w.$$

Under this, $L^1(\mathbb{C}^n)$ becomes an algebra. Like the ordinary convolution, twis
convolution also extends from $L^1(\mathbb{C}^n)$ to other $L^p(\mathbb{C}^n)$ and satisfies the Youn
inequality

$$\|f \times g\|_r \leqslant \|f\|_p \|g\|_q, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Though the twisted convolution is not commutative, it has better behaviour w
respect to L^p estimates. For example, we have the following.

Theorem 2.1. *For f and g in $L^2(\mathbb{C}^n)$, $f \times g$ is also in $L^2(\mathbb{C}^n)$ and*

$$\|f \times g\|_2 \leqslant (2\pi)^{n/2} \|f\|_2 \|g\|_2.$$

Further, we have $W(f \times g) = W(f) W(g)$.

A bounded operator $M \in \mathscr{B}(L^2(\mathbb{R}^n))$ is called a (left) Weyl multiplier of $L^p(\mathbb{C}^n)$ if
operator T_M defined on $f \in L^1 \cap L^p(\mathbb{C}^n)$ by $W(T_M f) = MW(f)$ extends to a boun
operator on $L^p(\mathbb{C}^n)$. We denote the Weyl multiplier class by M_W. The sp
$M_W(L^1(\mathbb{C}^n))$ is identified with $\mathscr{M}(\mathbb{C}^n)$, the Banach algebra of finite Borel measures
$\mathbb{C}^n$, and $M_W((L^2(\mathbb{C}^n))$ is the algebra $\mathscr{B}(L^2(\mathbb{R}^n)$ of all bounded operators on $L^2(\mathbb{R}^n)$.
any p, $1 < p < \infty$, a sufficient condition for L^p-Weyl multipliers has been proved
Mauceri in [5]. However, for the necessary part, only the following is known
Mauceri [5]).

Let $M \in M_W(L^p(\mathbb{C}^n))$, $1 \leqslant p < \infty$. Then there exists a tempered distribution $\rho \in S'($
such that for $f \in \mathscr{S}(\mathbb{C}^n)$, $T_M f = \rho \times f$.

In §3, we try to obtain such a characterization for $M \in M_W(L^p(\mathbb{C}^n))$, through eleme
in the dual space of a concrete function space, which we call pseudo measures.

Given a function f in $L^p(\mathbb{C}^n)$, $1 \leqslant p < \infty$, we have the special Hermite expansi
given by

$$f = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k. \qquad ($$

Here φ_k stands for the Laguerre function

$$\varphi_k(z) = L_k^{n-1}\left(\frac{1}{2}|z|^2\right)\exp(-|z|^2/4),$$

where L_k^{n-1} is the kth Laguerre polynomial of type $(n-1)$. For various results concerning the special Hermite expansions, we refer to [11].

Let L be the special Hermite operator defined by

$$L = -\Delta + \tfrac{1}{4}|z|^2 - i\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right).$$

Then the special Hermite functions are eigenfunctions of the operator L and the series (2.1) is the eigenfunction expansion associated to L. In view of this and spectral theorem one can define L^s(s real), by

$$L^s f = (2\pi)^{-n}\sum_{k=0}^{\infty}(2k+n)^s f \times \varphi_k.$$

We make use of these operators in the study of Weyl multipliers.

3. Weyl multipliers for $L^p(\mathbb{C}^n)$

Let $\mathscr{B} = \mathscr{B}(L^2(\mathbb{R}^n))$. We denote by $\mathscr{B}_2 = \mathscr{B}_2(L^2(\mathbb{R}^n))$, the Hilbert space of Hilbert–Schmidt operators on $(L^2(\mathbb{R}^n)$, with the norm $\|.\|_2$ and $\mathscr{B}_1 = \mathscr{B}_1(L^2(\mathbb{R}^n))$, the ideal of trace class operators. $\mathscr{B}_1$ is a Banach space under the norm $\|c\|_1 = \mathrm{tr}(|c|) = \mathrm{tr}(c^*c)^{1/2}$ and any element of $\mathscr{B}_1$ can be written as the product FG of two Hilbert–Schmidt operators $F.G$.

Let $A(\mathbb{C}^n)$ denote the space of function f on $\mathbb{C}^n$ whose Weyl transforms $W(f)$ are in B_1. Define

$$\|f\|_A = \|W(f)\|_1 \quad f\in A(\mathbb{C}^n).$$

Then $A(\mathbb{C}^n)$ is an algebra with the multiplication operation given by twisted convolution. Since any element of $\mathscr{B}_1$ is a product of two Hilbert–Schmidt operators and since any Hilbert–Schmidt operator is the Weyl transform of an L^2 function, $A(\mathbb{C}^n)$ contains precisely functions of the form $f \times g$ where f and g are from L^2. Thus $A(\mathbb{C}^n)$ is a subspace of $L^2(\mathbb{C}^n)$. It is easy to see that it is complete with the norm defined above. Thus $A(\mathbb{C}^n)$ is a Banach algebra under $\|.\|_A$, which also shows that Weyl transform is an isometric isomorphism of $A(\mathbb{C}^n)$ onto $\mathscr{B}_1$. We define $P(\mathbb{C}^n)$ to be the dual space of $A(\mathbb{C}^n)$. Then the adjoint of W, W^* will map $\mathscr{B}(L^2(\mathbb{R}^n))$ onto $P(\mathbb{C}^n)$. We call the elements of $P(\mathbb{C}^n)$, pseudo measures. Now, for $\sigma\in P(\mathbb{C}^n)$, $W(\sigma)$ is defined to be the unique element of $\mathscr{B}(L^2(\mathbb{R}^n))$ so that $W^*(W(\sigma)) = \sigma$. Thus, we have the following.

Theorem 3.1. *The Weyl transform $\sigma \mapsto W(\sigma)$ is an isometric isomorphism of $P(\mathbb{C}^n)$ onto $\mathscr{B}(L^2(\mathbb{R}^n))$.*

Let σ_1, $\sigma_2 \in P(\mathbb{C}^n)$. We define $\sigma_1 \times \sigma_2$ to be that pseudo measure for which $W(\sigma_1 \times \sigma_2) = W(\sigma_1)\,W(\sigma_2)$. This definition makes sense by the above theorem.

Theorem 3.2. *Let M be an L^p-multiplier for the Weyl transform. Then there exists a pseudo measure σ such that $T_M f = \sigma \times f$ for every $f\in L^1 \cap L^p(\mathbb{C}^n)$.*

Proof. As $M\in\mathscr{B}$, by theorem 3.1, there exists an element $\sigma\in P(\mathbb{C}^n)$ such that $W(\sigma)=M$. If h is a function in $L^p(\mathbb{C}^n)$, define, for each $g\in A(\mathbb{C}^n)$, $h(g)=\operatorname{tr}(W(h)W(g))$. Then

$$|h(g)|\leqslant \|W(h)\|\,\|W(g)\|_1=\|W(h)\|\,\|g\|_A,$$

which shows that h can be considered as an element of $P(\mathbb{C}^n)$. If $f\in L^1\cap L^p(\mathbb{C}^n)$, then the function $T_M f$ can be regarded as a pseudo measure for which the Weyl transform $W(T_M f)$ is defined as earlier. Now we claim that $W(T_M f)$, Weyl transform of the pseudo measure $T_M f$ coincide with the Weyl transform $W(T_M f)$ of the function $T_M f$. Let $P\in\mathscr{B}_1$. Then there exists a $g\in A(\mathbb{C}^n)$ such that $W(g)=P$. Thus we have

$$\begin{aligned}\langle P, W(T_M f)\rangle(\text{pseudo measure}) &= \langle W(g), W(T_M f)\rangle\\ &=\langle W^* W(g), T_M f\rangle\\ &=\langle g, T_M f\rangle\\ &=(T_M f)(g)\\ &=\operatorname{tr}(W(T_M f)W(g))\\ &=\operatorname{tr}(W(T_M f)P)\\ &=\langle P, W(T_M f)\rangle(\text{function}),\end{aligned}$$

as $W(T_M f)$ (function) belongs to M.

Thus $(W(T_M f))$, the Weyl transform of the pseudo measure $T_M f$ coincides with ordinary $W(T_M f)$ (Weyl transform of the function $T_M f$), which is precisely $MW(f)$. Again, as f can be considered as a pseudo measure, $\sigma\times f$ makes sense and $W(\sigma\times f)=W(\sigma)W(f)$. But $W(\sigma)=M$, from which it follows that $W(\sigma\times f)=W(T_M f)$, which in turn implies that $T_M f=\sigma\times f$. □

4. Laguerre Sobolev spaces

Let m be a positive integer. The Sobolev spaces $W_L^{m,p}(\mathbb{C}^n)$ are defined using certain vector on $\mathbb{C}^n$.

The special Hermite operator L can be written as

$$L=-\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j+\bar{Z}_jZ_j),$$

where the vector fields Z_j and $\bar{Z}_j$ on $\mathbb{C}^n$ are given by

$$Z_j=\frac{\partial}{\partial z_j}+\frac{1}{4}\bar{z}_j,\quad \bar{Z}_j=\frac{\partial}{\partial \bar{z}_j}-\frac{1}{4}z_j.$$

For $m\geqslant 1$, an integer, we define $W_L^{m,p}(\mathbb{C}^n)$ to be the collection of those functions f in $L^p(\mathbb{C}^n)$ for which $Z^\alpha\bar{Z}^\beta\in L^p(\mathbb{C}^n)$, $|\alpha|+|\beta|\leqslant m$. Here

$$Z^\alpha=Z_1^{\alpha_1}Z_2^{\alpha_2}\cdots Z_n^{\alpha_n},\quad \bar{Z}^\beta=\bar{Z}_1^{\beta_1}\bar{Z}_2^{\beta_2}\cdots\bar{Z}_n^{\beta_n},$$

$\alpha.\beta\in\mathbb{N}^n$. When m is an even integer it follows that $L^{m/2}f\in L^p(\mathbb{C}^n)$ whenever $f\in W_L^{m,p}(\mathbb{C}^n)$. Now if we define

$$\|f\|_{W_L^{m,p}}=\sum_{|\alpha|+|\beta|\leqslant m}\|Z^\alpha\bar{Z}^\beta f\|_p,$$

then $W_L^{m,p}$ turns out to be a Banach space under $\|.\|_{W_L^{m,p}}$.

Let $D(W_L^{1,p})(\mathbb{C}^n)$ denote the collection of functions of the form

$$f = f_0 + \sum_{j=1}^{n} Z_j f_j + \sum_{j=1}^{n} \bar{Z}_j g_j, \ldots,$$

where $f_0, f_j \cdot g_j \in W_L^{1,p}(\mathbb{C}^n)$ for $j = 1, 2, \ldots, n$. Then $D(W_L^{1,p})(\mathbb{C}^n)$ becomes a Banach space if we define the norm $\|.\|_D$ as follows:

$$\|f\|_D = \inf\left\{ \max_{j=1 \text{ to } n} (\|f_0\|_{W_L^{1,p}}, \|f_j\|_{W_L^{1,p}}, \|g_j\|_{W_L^{1,p}}) \right\},$$

where the infimum is taken over all representations of f in the above form. Clearly $W_L^{1,p}(\mathbb{C}^n)$ is contained in $D(W_L^{1,p})(\mathbb{C}^n)$, which in turn is contained in $L^p(\mathbb{C}^n)$. In Proposition 4.1 we will actually show that $D(W_L^{1,p})(\mathbb{C}^n) = \mathrm{L}^p(\mathbb{C}^n)$.

Given a bounded operator M on $L^2(\mathbb{R}^n)$, we can define an operator T_M on $L^2 \cap (W_L^{1,p})(\mathbb{C}^n)$ by $W(T_M f) = MW(f)$. We say that M is a left $W_L^{1,p}$ multiplier for the Weyl transform if T_M extends to a bounded operator on $(W_L^{1,p})(\mathbb{C}^n)$.

We first prove the following result.

Theorem 4.1. *Let $1 \leqslant p < \infty$. Then we have the following:*

$$M_W(W_L^{1,p}) = M_W(D(W_L^{1,p})) = M_W(D(W_L^{1,p}), L^p).$$

Proof. As $D(W_L^{1,p}) \subset L^p$, we get

$$M_W(D(W_L^{1,p}), D(W_L^{1,p})) \subset M_W(D(W_L^{1,p}), L^p). \tag{4.1}$$

Suppose $M \in M_W(D(W_L^{1,p}), L^p)$. Let $f \in W_L^{1,p}$. Define T_M on $W_L^{1,p} \cap L^2(\mathbb{C}^n)$ by $W(T_M f) = MW(f)$. As $W_L^{1,p} \subset D(W_L^{1,p})$, $T_M f \in L^p$. Let $A_j = -(\partial/\partial x_j) + x_j$, $A_j^* = (\partial/\partial x_j) + x_j$. Then

$$W(Z_j T_M f) = iW(T_M f)A_j = MiW(f)A_j = W(T_M Z_j f) \tag{4.2}$$

and

$$W(\bar{Z}_j T_M f) = iW(T_M f)A_j^* = W(T_M \bar{Z}_j f). \tag{4.3}$$

Further, as $f \in W_L^{1,p}$, $Z_j f$, $\bar{Z}_j f \in D(W_L^{1,p})$, and so $T_M Z_j f$, $T_M \bar{Z}_j f \in L^p$. Thus it follows from (4.2) and (4.3), that $Z_j T_M f$, $\bar{Z}_j T_M f \in L^p$, which will then imply that $T_M f \in W_L^{1,p}$. By definition,

$$\|T_M f\|_{W_L^{1,p}} = \|T_M f\|_p + \sum_{j=1}^{n} \|Z_j T_M f\|_p + \sum_{j=1}^{n} \|\bar{Z}_j T_M f\|_p.$$

But

$$\|T_M f\|_p \leqslant C_M \|f\|_D \leqslant C_M \|f\|_{W_L^{1,p}},$$

$$\|Z_j T_M f\|_p = \|T_M Z_j f\|_p \leqslant C_M \|Z_j f\|_D \leqslant C_M \|f\|_{W_L^{1,p}}$$

and

$$\|\bar{Z}_j T_M f\|_p \leqslant C_M \|f\|_{W_L^{1,p}}.$$

Thus

$$\|T_M f\|_p \leqslant (2n+1) C_M \|f\|_{W_L^{1,p}},$$

which shows that T_M is a bounded operator on $W_L^{1,p}(\mathbb{C}^n)$. Hence

$$M_W(D(W_L^{1,p}), L^p) \subset M_W(W_L^{1,p}). \tag{4.4}$$

Now let $M \in M_W(W_L^{1,p})$. For $f \in D(W_L^{1,p})$, we define

$$\tilde{T}_M f = T_M f_0 + \sum_{j=1}^{n} Z_j T_M f_j + \sum_{j=1}^{n} \bar{Z}_j T_M g_j.$$

To prove $\tilde{T}_M$ is well defined, assume that $f \in D(W_L^{1,p})$ is a representation of 0, viz

$$f_0 + \sum_{j=1}^{n} Z_j f_j + \sum_{j=1}^{n} \bar{Z}_j g_j = 0, \quad f_0, f_j, g_j \in W_L^{1,p}, 1 \leqslant j \leqslant n.$$

Consider

$$\begin{aligned} W(\tilde{T}_M f) &= W(T_M f_0) + \sum_{j=1}^{n} W(Z_j T_M f_j) + \sum_{j=1}^{n} W(\bar{Z}_j T_M g_j) \\ &= M W(f_0) + \sum_{j=1}^{n} iMW(f_j) A_j + \sum_{j=1}^{n} iMW(g_j) A_j^* \\ &= M W\left(f_0 + \sum_{j=1}^{n} Z_j f_j + \sum_{j=1}^{n} \bar{Z}_j g_j\right) \\ &= 0. \end{aligned}$$

Thus $\tilde{T}_M$ is well defined and $W(\tilde{T}_M f) = M W(f)$. The proof will be complete if we could show that $\tilde{T}_M$ is bounded. Let $f \in D(W_L^{1,p})$. Consider

$$\begin{aligned} \|\tilde{T}_M f\|_D &\leqslant \max_{1 \leqslant j \leqslant n} (\|T_M f_0\|_{W_L^{1,p}}, \|T_M f_j\|_{W_L^{1,p}}, \|T_M g_j\|_{W_L^{1,p}}) \\ &\leqslant C_M \max_{1 \leqslant j \leqslant n} (\|f_0\|_{W_L^{1,p}}, \|f_j\|_{W_L^{1,p}}, \|g_j\|_{W_L^{1,p}}) \end{aligned}$$

which is true for any representation $f_0 + \sum_{j=1}^{n} Z_j f_j + \sum_{j=1}^{n} \bar{Z}_j g_j$ of f. Hence it follows that $\|\tilde{T}_M f\|_D \leqslant C_M \|f\|_D$, which shows that $\tilde{T}_M$ is a bounded operator on $D(W_L^{1,p})$. Thus

$$M_W(W_L^{1,p}) \subset M_W(D(W_L^{1,p})). \tag{4.5}$$

From (4.1), (4.4) and (4.5) we get the required result. □

For $m \in \mathbb{N}$, $D(W_L^{m,p})$ is defined as earlier, viz

$$D(W_L^{m,p}) = \left\{ f = f_0 + \sum_{j=1}^{n} Z_j f_j + \sum_{j=1}^{n} \bar{Z}_j g_j \right\},$$

where $f_0, f_j, g_j \in W_L^{m,p}(\mathbb{C}^n)$, $j = 1, 2, \ldots, n$. With this definition, we have the following.

Theorem 4.2. *Let $1 \leqslant p < \infty$ and m, an integer $\geqslant 1$. Then we have the following*

$$M_W(W_L^{m,p}) = M_W(D(W_L^{m,p})) = M_W(D(W_L^{m,p}), W_L^{m-1,p}).$$

PROPOSITION 4.1

Let $1 < p < \infty$. Then $D(W_L^{1,p}) = L^p$.

Proof. Let $f \in L^p$. Write $f = LL^{-1} f$, viz

$$f = \sum_{j=1}^{n} Z_j \left(\frac{1}{2} \bar{Z}_j L^{-1} f \right) + \sum_{j=1}^{n} \bar{Z}_j \left(\frac{1}{2} Z_j L^{-1} f \right).$$

We claim that $\bar{Z}_j L^{-1} f$ and $Z_j L^{-1} f$ are in $W_L^{1,p}(\mathbb{C}^n)$. In theorem 2.2.2 of [11] it has been proved that $\bar{Z}_j L^{-1/2}$ and $Z_j L^{-1/2}$ are bounded operators on $L^p(\mathbb{C}^n)$, $1 < p < \infty$. The same argument shows that $\bar{Z}_j L^{-1}$ and $Z_j L^{-1}$ are also bounded on $L^p(\mathbb{C}^n)$ (see the reasoning below).

Let now $S_j f = Z_j \bar{Z}_j L^{-1} f$, $S_j^* f = \bar{Z}_j Z_j L^{-1} f$. We claim that $S_j f, S_j^* f \in L^p$. In view of Theorem 2.2.1 of [11], we have to show that S_j and S_j^* are twisted convolution operators with Calderon–Zygmund kernels and they are bounded on L^2. Consider the operator S_j. We can write

$$S_j f = -(2\pi)^{-n/2} \sum_{\mu} \left(\frac{2\mu_j + 2}{2|\mu| + n} \right) f \times \phi_{\mu\mu}, \tag{4.6}$$

as $Z_j(\phi_{\mu\nu}) = i(2\nu_j)^{1/2}\, \phi_{\mu,\nu-\varepsilon_j}$, $\bar{Z}_j(\phi_{\mu\nu}) = i(2\nu_j + 2)^{1/2}\, \phi_{\mu,\nu+\varepsilon_j}$ and $L(\phi_{\mu\nu}) = (2|\nu| + n)\phi_{\mu\nu}$. From (4.6), it is clear that S_j is bounded on L^2. And S_j is given by $S_j f = f \times k_j$ where

$$k_j = Z_j \bar{Z}_j \int_0^\infty k_t(z)\,\mathrm{d}t$$

and $k_t(z)$ is the kernel of $\exp(-t\,L)$ given by

$$k_t(z) = (\sinh 2t)^{-n} \exp(-\coth t |z|^2).$$

We can show that k_j satisfies

$$|k_j(z)| \leqslant c|z|^{-2n},$$
$$|\nabla k_j(z)| \leqslant c|z|^{-2n-1}.$$

Thus, from theorem 2.2.1 of [11], we conclude that S_j is bounded on L^p. Similarly we can show that S_j^* is bounded on L^p. Then, it follows that $\bar{Z}_j L^{-1} f, Z_j L^{-1} f \in W_L^{1,p}$, which shows that $f \in D(W_L^{1,p})$.

We can also prove the following.

PROPOSITION 4.2

Let $1 < p < \infty$. Then $D(W_L^{m,p}) = W_L^{m-1,p}$ for $m \geqslant 1$ any integer.

Putting the above facts together, we obtain the following.

Theorem 4.3. *Let $1 < p < \infty$ and m, any integer $\geqslant 1$. Then the space of Weyl multipliers for the Laguerre Sobolev space $W_L^{m,p}(\mathbb{C}^n)$ coincide with the space of Weyl multipliers for $L^p(\mathbb{C}^n)$.*

This, combined with the theorem 3.2, leads to the following Corollary.

COROLLARY 4.1

Let $1 < p < \infty$ and m, any integer $\geqslant 1$. Let M be a Weyl multiplier for the Laguerre Sobolev space $W_L^{m,p}(\mathbb{C}^n)$. Then there exists a pseudo measure σ such that $T_M f = \sigma \times f \quad \forall f \in C_c^\infty(\mathbb{C}^n)$.

5. The space $M_W(W_L^{m,1}(\mathbb{C}^n))$

We first remark that $M_W(L^1(\mathbb{C}^n)) \subset M_W(W_L^{1,1}(\mathbb{C}^n))$. For, suppose $M \in M_W(L^1(\mathbb{C}^n))$, define T_M on $L^2 \cap (W_L^{1,1}(\mathbb{C}^n))$ by $W(T_M f) = MW(f)$. Let $f \in W_L^{1,1}$. Then $Z_j f$,

$\bar{Z}_j f \in L^1$. Therefore, it is easy to verify that $W(T_M Z_j f) = W(Z_j T_M f)$ and $W(T_M \bar{Z}_j f) = W(\bar{Z}_j T_M f)$. Then, as $M \in M_W(L^1, L^1)$, it follows that $T_M f \in W_L^{1,1}$ and $\| T_M f \|_{W_L^{1,1}} \leqslant C_M \| f \|_{W_L^{1,1}}$ for every $f \in W_L^{1,1}$.

Let S denote the collection of elements f of the form $f = \sum f_i \times g_i$, where $f_i \in D(W_L^{1,1})$, $g_i \in C_0(\mathbb{C}^n)$, $\sum \| f_i \|_D \| g_i \|_\infty < \infty$. Then S is a Banach space under the norm

$$\| f \|_s = \inf\left(\sum \| f_i \|_D \| g_i \|_\infty \right),$$

where the infimum is taken over all representations of f in the above form. Then we prove the following theorem.

Theorem 5.1. *There is an isometric isomorphism of $M_W(W_L^{1,1})$ onto the dual space S^* of S.*

Proof. By theorem 4.1, we have $M_W(W_L^{1,1}) = M_W(D(W_L^{1,1}), L^1)$. Suppose $M \in M_W(D(W_L^{1,1}), L^1)$. For $f = \sum f_i \times g_i \in S$, define $\beta_M(f) = \sum_i T_M f_i \times g_i(0)$. As $T_M f_i \in L^1$ and $g_i \in C_0$, $T_M f_i \times g_i \in C_0$ and $T_M f_i \times g_i(0)$ is meaningful. To prove β_M is well defined, let $f = \sum f_i \times g_i$ be a representation of 0. Choose an approximate identity $\{e_\alpha\} \subset C_c^\infty(\mathbb{C}^n)$ for $D(W_L^{1,1})$ such that $\| e_\alpha \|_1 \leqslant 1$. As

$$\begin{aligned} \| T_M(e_\alpha \times f_i) \times g_i - T_M f_i \times g_i \|_\infty &\leqslant \| T_M(e_\alpha \times f_i) - T_M f_i \|_1 \| g_i \|_\infty \\ &\leqslant \| T_M \| \| e_\alpha \times f_i - f_i \|_D \| g_i \|_\infty, \end{aligned}$$

left hand side tends to 0 as $\alpha \to \infty$. Further,

$$\begin{aligned} \| \sum_i T_M(e_\alpha \times f_i) \times g_i(0) \| &\leqslant \| T_M \| \sum_i \| e_\alpha \times f_i \|_D \| g_i \|_\infty \\ &\leqslant \| T_M \| \sum_i \| f_i \|_D \| g_i \|_\infty \ (\text{as } \| e_\alpha \|_1 \leqslant 1), \end{aligned}$$

which shows that $\sum_i T_M(e_\alpha \times f_i) \times g_i(0)$ converges to $\sum_i T_M f_i \times g_i(0)$. Now for each α,

$$\sum_i T_M(e_\alpha \times f_i) \times g_i(0) = (T_M e_\alpha \times \sum_i f_i \times g_i)(0) = 0.$$

Thus $\sum_i T_M f_i \times g_i(0) = 0$, proving that β_M is well defined. β_M satisfies

$$| \beta_M(f) | \leqslant \| T_M \| \sum_i \| f_i \|_D \| g_i \|_\infty,$$

which is true for every representation $\sum f_i \times g_i$ of f, showing that $| \beta_M(f) | \leqslant \| T_M \| \| f \|_S$ or

$$\| \beta_M \| s^* \leqslant \| T_M \|. \tag{5.1}$$

On the other hand,

$$\| T_M \| = \sup_{\| f \|_D \leqslant 1} \| T_M f \|, \quad (f \in D(W_L^{1,1}))$$

and

$$\| T_M f \|_1 = \| T_M f \| C_0^* = \sup_{\| g \|_\infty \leqslant 1} | T_M f(g) | \quad (g \in C_0).$$

But

$$| T_M f(g) | = | T_M f \times g(0) | = | \beta_M(f \times g) | \leqslant \| \beta_M \| S^* \| f \|_D \| g \|_\infty,$$

from which it follows that

$$\| T_M \| \leqslant \| \beta_M \| S^*. \tag{5.2}$$

From (5.1) and (5.2), we see that $T_M \mapsto \beta_{T_M}$ is an isometry. To prove the mapping is surjective, let us assume that $\beta \in S^*$. Fix $f \in D(W_L^{1,1})$, define for each $g \in C_0$, $F_f(g) = \beta(f \times g)$. Then

$$|F_f(g)| = |\beta(f \times g)| \leqslant \|\beta\|S^* \|f\|_D \|g\|_\infty,$$

which shows that F_f is a continuous linear functional on C_0. Hence there exists a unique $\mu_f \in M(\mathbb{C}^n)$ such that

$$F_f(g) = \beta(f \times g) = \mu_f(g) = \mu_f \times g(0). \tag{5.3}$$

Choose an approximate identity $\{e_\alpha\} \subset C_c^\infty(\mathbb{C}^n)$ for $D(W_L^{1,1})$. Then corresponding to each e_α, we have a unique μ_{e_α} in $M(\mathbb{C}^n)$ satisfying (5.3). Since $\mu_{e_\alpha} \in M(\mathbb{C}^n) = M_W(L^1(\mathbb{C}^n))$, μ_{e_α} is identified with $M_{e_e} = M_\alpha \in M_W(L^1(\mathbb{C}^n))$ such that

$$T_{M_\alpha} f = \mu_{e_\alpha} \times f \quad \forall f \in L^1(\mathbb{C}^n). \tag{5.4}$$

As $\{e_\alpha\}$ is an approximate identity for $D(W_L^{1,1})$, we get $\|e_\alpha \times h \times g - h \times g\|_D \to 0$ as $\alpha \to \infty$. Then as $\beta \in S^*$, we have

$$\lim_\alpha \beta(e_\alpha \times h \times g) = \beta(h \times g).$$

Thus, it follows from (5.3) that $\lim_\alpha \mu_{e_\alpha}(h \times g)$ exists for every $h \in C_c^\infty(\mathbb{C}^n)$ and $g \in C_0(\mathbb{C}^n)$. As the collection of elements $h \times g$, $h \in C_c^\infty(\mathbb{C}^n)$, $g \in C_0(\mathbb{C}^n)$, is dense in $C_0(\mathbb{C}^n)$, we have $\lim_\alpha \mu_{e_\alpha}(g)$ exists for every $g \in C_0(\mathbb{C}^n)$. There exists a $\mu \in M(\mathbb{C}^n)$ such that μ_{e_α} converges to μ in the weak * topology. Also $\|\mu\| = \lim_\alpha \|\mu_{e_\alpha}\|$. As $\mu \in M(\mathbb{C}^n)$, μ is identified with an operator $M \in M_W(L^1(\mathbb{C}^n))$ such that $T_M f = \mu \times f$ for every $f \in L^1(\mathbb{C}^n)$. As μ_{e_α} converges to μ, it follows from (5.4) that $\lim_\alpha T_{M_\alpha} f = T_M f$ for every $f \in L^1(\mathbb{C}^n)$. By the remark mentioned earlier, we get $M \in M_W(W_L^{1,1}) = M_W(D(W_L^{1,1}).L^1)$. The proof will be complete if we could show that β_M coincides with β on S. For $h \in C_c^\infty(\mathbb{C}^n)$, $g \in C_0(\mathbb{C}^n)$, we have

$$\begin{aligned}
\beta_M(h \times g) &= \lim_\alpha T_{M_\alpha} h \times g(0) \\
&= \lim_\alpha \mu_{e_\alpha} \times h \times g(0) \\
&= \lim_\alpha \beta(e_\alpha \times h \times g) \\
&= \beta(h \times g).
\end{aligned}$$

Using the density argument, we can show that $\beta_M(f \times g) = \beta(f \times g)$ for every $f \in D(W_L^{1,1})$, $g \in C_0$, thus proving our assertion.

If we define S_m ($m \in \mathbb{N}$) to be the collection of elements f of the form $f = \sum f_i \times g_i$, where $f_i \in D(W_L^{m,1})$, $g_i \in C_0$, $\sum \|f_i\|_D \|g_i\|_\infty < \infty$. Then S_m becomes a Banach space. Using the facts that $\|f\|_1 \leqslant \|f\|_D$ for every $f \in D$, $M_W(L^1) \subset M_W(W_L^{m,1})$ and by making use of the proof of Theorem 5.1,we obtain the following.

Theorem 5.2. *Let m be an integer such that $m \geqslant 1$. Then there is a continuous isomorphism of $M_W(W_L^{m,1})$ onto the dual space S_m^* of S_m.*

Acknowledgement

The authors are grateful to the referee for his careful reading of the manuscript and for making many useful suggestions. One of the authors (RR) thanks the Council of Scientific and Industrial Research for financial support.

References

[1] Folland G B, Harmonic analysis in phase space. *Ann. Math. Stud.* 112 (Princeton University Press) (1989)

[2] Gaudry G I, Quasi measures and operators commuting with convolution. *Pacific J. Math* **18** (1966) 461–476

[3] Hormander L, Estimates for translation invariant operators in L^p spaces, *Acta Math.* **104** (1960) 93–140

[4] Larsen R, *An introduction to the theory of multipliers* (Berlin: Springer Verlag) (1971)

[5] Mauceri G, The Weyl transform and bounded operators on $L^p(\mathbb{R}^n)$, *J. Funct. Anal.* **39** (1980) 408–429

[6] Peetre J and Sparr G, Interpolation and non-commutative integration, *Annali di Mat. Pura ed Applicata* **CIV** (1975) 187–207

[7] Poornima S, Multipliers of Sobolev spaces, *J. Funct. Anal.* **45** (1982) 1–28

[8] Stein E M, *Singular integrals and differentiability properties of functions* (Princeton University Press) (1972)

[9] Thangavelu S, Spherical means on the Heisenberg group and a restriction theorem for the symplectic Fourier transform, *Revist Mat. Ibero.* **7** (1991) 135–155

[10] Thangavelu S, On regularity of twisted spherical means and special Hermite expansions, *Proc. Indian Acad. Sci.* **103** (1993) 303–320

[11] Thangavelu S, *Lectures on Hermite and Laguerre expansions,* Mathematical Notes 42 (Princeton University Press) (1993)

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 1, February 1998, pp. 41–49.

On the neutrix convolution product of $x_-^r \ln x_-$ and x_+^{-s}

EMIN ÖZÇAĞ

Department of Mathematics, University of Hacettepe, 06532-Beytepe, Ankara, Turkey
E-mail: ozcag1@eti.cc.hun.edu.tr

MS received 12 September 1997; revised 18 November 1997

Abstract. The existence of the neutrix convolution product of distribution $x_-^r \ln x_-$ and x_+^{-s} is proved and some convolution products are evaluated.

Keywords. Distribution; neutrix; neutrix limit; neutrix convolution product.

Let $\mathscr{D}$ be the space of infinitely differentiable functions on the real line R with compact support and $\mathscr{D}'$ be the space of distributions defined on $\mathscr{D}$. The convolution product $f * g$ of two distributions f and g in $\mathscr{D}'$ may be defined by

$$\langle (f * g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x+y) \rangle \rangle \tag{1}$$

for arbitrary ϕ in $\mathscr{D}$ if f and g satisfy either of the conditions:

(B1) eiher f or g has bounded support,
(B2) the supports of f and g are bounded on the same side, (see [5]).

It follows from the definition that if the convolution products $f * g$ and $f * g'$ (or $f' * g$) exist then

$$f * g = g * f,$$
$$(f * g)' = f * g' = f' * g. \tag{2}$$

Now let f and g be two distributions on R such that $A = \mathrm{Supp}\,(f)$ and $B = \mathrm{Supp}\,(g)$ satisfying the following conditions:

(i) for every bounded set $K \subset R$, the set $(A \times B) \cap K^{\Delta}$ is bounded in R^2,
(ii) for every bounded set $K \subset R$, the set $A \cap (K - B)$ is bounded in R,
(iii) for every bounded set $K \subset R$, the set $(K - A) \cap B$ is bounded in R,
(iv) if $x_n \in A$, $y_n \in B$ and $|x_n| + |y_n| \to \infty$, then $|x_n + y_n| \to \infty$,

where $K^{\Delta} = \{(x, y) \in R^2 : x + y \in K\}$, then the convolution product $f * g$ of f and g exists and is defined as in (1).

The conditions (i)–(v) are well known (see [4, 7]). Condition (iv) was introduced by Mikusinski in [1]. If the supports of distributions f and g satisfy conditions (B1) or (B2), then they fulfill conditions (i)–(iv). In [4], Fisher and Kaminski gave two pairs of distributions F, G and f, g which did not satisfy the conditions (B1) and (B2), but the convolution products $F * G$ and $f * g$ exist and the conditions (i)–(iv) were satisfied.

The convolution product of distributions may be defined in a more general way without any restriction on the supports. The most known are given by Vladimirov and

Jones (see [8, 10]). However, there still exist many pairs of distributions such that convolution products do not exist in the sense of these definitions. In order to ex the convolution product to larger class of distributions Fisher developed the met which is very similar to Hadamard's method used to define the pseudo-functi regarded as a particular application of the so called neutrix calculus developed by der Corput [2]. His method is based on discarding of unwanted infinite quantities f asymptotic expansions and has been widely exploited in connection with the prob of distributional multiplication and convolution, see [6]. To introduce Fisher's de tion of convolution product, we need the following.

Let f and g be distributions and let $f_n = f\tau_n$ $(n = 1, 2, \dots)$ where τ is an infin differentiable function satisfying the following conditions:

(a) $\tau(x) = \tau(-x)$,
(b) $0 \leqslant \tau(x) \leqslant 1$,
(c) $\tau(x) = 1$ for $|x| \leqslant \frac{1}{2}$,
(d) $\tau(x) = 0$ for $|x| \geqslant 1$,

and the function τ_n is defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leqslant n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \dots$.

Then the neutrix convolution product $f \circledast g$ is defined as the neutrix limit of sequence $\{f_n * g\}$, providing the limit h exists in the sense that

$$N - \lim_{n \to \infty} \langle f_n * g, \phi \rangle = \langle h, \phi \rangle,$$

for all ϕ in $\mathscr{D}$, where N is the neutrix (see [2]), having domain $N' = \{1, 2, \dots, n,$ and range N'' the real numbers, with negligible functions finite linear sums of functions

$$n^\lambda \ln^{r-1} n, \ \ln^r n \ (\lambda > 0, r = 1, 2, \dots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In this definition the convolution product $f_n * g$ exists since the distribution having bounded support. Because the definition in nonsymmetric, the convolu product $f \circledast g$ is not always commutative.

The essential use of the neutrix limit is to extract the finite part from a diver quantity. In the neutrix calculus each limit, if properly defined, always exists. neutrix convolution product depends on the negligible functions and so depends on sequence τ_n. The negligible functions in the neutrix N given above are selected bec these are the functions that occur in mathematics and physics.

In [3], Fisher proved that his definition was a generalization of the convolu product in the sense of Gel'fand and Shilov. Indeed, if f and g be distributions i satisfying either conditions (B1) or (B2), then the neutrix convolution product exists and

$$f \circledast g = f * g.$$

He also proved that if the neutrix convolution product $f \circledast g'$ exists, then

$$(f \circledast g)' = f * g'. \tag{3}$$

Note however that equation (2) does not necessarily hold for the neutrix convolution product and that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$. In this paper, using Fisher's definition we get some results. First of all we give the following theorem.

Theorem 1. *The neutrix convolution product* $(x_-^r \ln x_-) \circledast \ln x_+$ *exists and*

$$\begin{aligned}
&(x_-^r \ln x_-) \circledast \ln x_+ \\
&= -\frac{x_-^{r+1}\ln^2 x_-}{2(r+1)} + \frac{(-1)^{r+1}}{2(r+1)} x_+^{r+1}\ln^2 x_+ - \frac{(-1)^{r+1}}{(r+1)^2} x_+^{r+1}\ln x_+ \\
&\quad + \frac{\psi(r+1)}{r+1} x_-^{r+1}\ln x_- + \frac{(-1)^{r+1}\xi(2)}{r+1} x_+^{r+1} + \frac{(-1)^{r+1}}{(r+1)^3} x_+^{r+1} \\
&\quad - \left[\frac{\psi(r+1)}{r+1} + \xi(2) + \sum_{i=1}^{r} i^{-2}\right]\frac{x_-^{r+1}}{r+1}
\end{aligned} \tag{4}$$

for $r = 1, 2, \ldots$ where

$$\psi(r) = \begin{cases} 0, & r = 0 \\ \Sigma_{i=1}^r i^{-1}, & r \geq 1. \end{cases}$$

$$\xi(2) = \sum_{i=1}^{\infty} i^{-2}.$$

Proof. Putting $(x_-^r \ln x_-)_n = (x_-^r \ln x_-)\tau_n(x)$, we have

$$\begin{aligned}
&\langle (x_-^r \ln x_-)_n * \ln x_+, \phi(x)\rangle = \langle (y_-^r \ln y_-)_n, \langle \ln x_+, \phi(x+y)\rangle\rangle \\
&= \int_{-n-n^{-n}}^{0} (-y)^r \ln(-y)\tau_n(y) \int_a^b \ln(x-y)_+ \phi(x)\,dx\,dy \\
&= \int_a^b \phi(x) \int_{-n}^{0} (-y)^r \ln(-y)\ln(x-y)_+\,dy\,dx \\
&\quad + \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^r \ln(-y)\tau_n(y)\ln(x-y)\,dy\,dx
\end{aligned} \tag{5}$$

for $n > -a$ and arbitrary ϕ in $\mathscr{D}$ with compact support contained in $[a, b]$. When $x < 0$, we have on making the substitution $y = xu^{-1}$,

$$\begin{aligned}
&\int_{-n}^{o} (-y)^r \ln(-y)\ln(x-y)_+\,dy = \int_{-n}^{x} (-y)^r \ln(-y)\ln(x-y)\,dy \\
&\quad = (-x)^{r+1}\int_{-x/n}^{1} u^{-r-2}[\ln^2(-x) + \ln(-x)\ln(1-u) - 2\ln(-x)\ln u \\
&\qquad - 2\ln u \ln(1-u) + \ln^2 u]\,du \\
&\quad = I_{1n} + I_{2n} - I_{3n} - I_{4n} + I_{5n}.
\end{aligned} \tag{6}$$

In [9], the neutrix limits of $I_{1n}, I_{2n}, I_{3n}, I_{4n}$ and I_{5n} were evaluated as

$$N-\lim_{n\to\infty} I_{1n} = -\frac{(-x)^{r+1}\ln^2(-x)}{r+1},$$

$$N-\lim_{n\to\infty} I_{2n} = -\frac{(-x)^{r+1}\ln(-x)}{(r+1)^2} + \frac{(-x)^{r+1}\ln^2(-x)}{r+1} + \frac{\psi(r)}{r+1}(-x)^{r+1}\ln(-x),$$

$$N-\lim_{n\to\infty} I_{3n} = -\frac{2(-x)^{r+1}\ln(-x)}{(r+1)^2},$$

$$N-\lim_{n\to\infty} I_{4n} = -\frac{(-x)^{r+1}}{(r+1)^3} + \frac{\xi(2)(-x)^{r+1}}{r+1} + \frac{\psi(r)(-x)^{r+1}}{(r+1)^2} + \frac{(-x)^{r+1}\ln^2(-x)}{2(r+1)} + \frac{1}{r+1}\sum_{i=1}^{r} i^{-2}(-x)^{r+1},$$

$$N-\lim_{n\to\infty} I_{5n} = -\frac{2(-x)^{r+1}}{(r+1)^3}.$$

It follows that

$$N-\lim_{n\to\infty}\int_{-n}^{0}(-y)^r\ln(-y)\ln(x-y)_+\,\mathrm{d}y = -\frac{x_-^{r+1}\ln^2 x_-}{2(r+1)} + \frac{\psi(r+1)}{r+1}x_-^{r+1}\ln x_- - \left[\frac{\psi(r+1)}{r+1} + \xi(2) + \sum_{i=1}^{r} i^{-2}\right]\frac{x_-^{r+1}}{r+1}. \tag{7}$$

When $x>0$, we have on making the substitution $y = x(1-u^{-1})$,

$$\begin{aligned}\int_{-n}^{0}(-y)^r\ln(-y)\ln(x-y)_+\,\mathrm{d}y &= x^{r+1}\int_{x/(x+n)}^{1} u^{-r-2}[\ln^2 x - 2\ln x\ln u + \ln x\ln(1-u) - \ln u\ln(1-u) \\ &\qquad + \ln^2 u](1-u)^r\,\mathrm{d}u \\ &= J_{1n} - J_{2n} + J_{3n} - J_{4n} + J_{5n}.\end{aligned} \tag{8}$$

Since

$$\begin{aligned}\int_{x/(x+n)}^{1} u^{-r-2}(1-u)^r\,\mathrm{d}u &= \sum_{i=1}^{r}(-1)^i\binom{r}{i}\int_{x/(x+n)}^{1} u^{-r+i-2}\,\mathrm{d}u \\ &= \sum_{i=1}^{r}\frac{(-1)^i}{r-i+1}\binom{r}{i}[(1+n/x)^{r-i+1}-1],\end{aligned}$$

it follows that

$$N-\lim_{n\to\infty} J_{1n} = 0. \tag{9}$$

Now, integrating by parts we have

$$\int_{x/(x+n)}^{1} u^{-r-2}\ln u(1-u)^r \mathrm{d}u = -\frac{1}{r+1}\int_{x/(x+n)}^{1} (1-u)^r \ln u \mathrm{d}(u^{-r-1})$$

$$= \frac{(x+n)[\ln x - \ln n - \ln(1+x/n)]n^r}{(r+1)x^{r+1}}$$

$$+\frac{1}{r+1}\int_{x/(x+n)}^{1} u^{-r-2}(1-u)^r \mathrm{d}u - \frac{r}{r+1}\int_{x/(x+n)}^{1} u^{-r-1}(1-u)^{r-1}\ln u \mathrm{d}u.$$

It can be easily seen that

$$N-\lim_{n\to\infty}\frac{(x+n)[\ln x - \ln n - \ln(1+x/n)]n^r}{(r+1)x^{r+1}} = \frac{(-1)^r}{r+1}\left(\frac{1}{r}-\frac{1}{r+1}\right).$$

If we assume that

$$a_{r-1} = N-\lim_{n\to\infty}\int_{x/(x+n)}^{1} u^{-r-1}(1-u)^{r-1}\ln u \mathrm{d}u.$$

exits, it follows that a_r exists and

$$(r+1)a_r + ra_{r-1} = (-1)^r\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

for $r = 1, 2, \ldots$. Since

$$a_0 = N-\lim_{n\to\infty}\int_{x/(x+n)}^{1} u^{-2}\ln u \mathrm{d}u = \ln x - 1$$

certainly exists, it follows easily by induction that

$$a_r = \frac{(-1)^r}{r+1}\ln x - \frac{(-1)^r}{(r+1)^2}.$$

Thus

$$N-\lim_{n\to\infty}(-J_{2n}) = \frac{2(-1)^{r+1}}{r+1}x^{r+1}\ln^2 x - \frac{2(-1)^{r+1}}{(r+1)^2}x^{r+1}\ln x. \tag{10}$$

Next

$$\int_{x/(x+n)}^{1} u^{-r-2}\ln(1-u)(1-u)^r \mathrm{d}u = -\frac{(x+n)n^r}{(r+1)x^{r+1}}\ln(1+x/n)$$

$$-\frac{1}{r+1}\int_{x/(x+n)}^{1} u^{-r-1}(1-u)^{r-1}\mathrm{d}u - \frac{r}{r+1}\int_{x/(x+n)}^{1} u^{-r-1}\ln(1-u)(1-u)^{r-1}\mathrm{d}u.$$

Assuming that

$$b_{r-1} = N-\lim_{n\to\infty}\int_{x/(x+n)}^{1} u^{-r-1}(1-u)^{r-1}\ln(1-u)\mathrm{d}u$$

exits, we see with the same argument as above that b_r exists and

$$(r+1)b_r + rb_{r-1} = \frac{(-1)^r}{r+1}\left(\frac{1}{r}-\frac{1}{r+1}\right).$$

Since $b_0 = \ln x - 1$, it follows that

$$b_r = \frac{(-1)^r}{r+1}\ln x - \frac{(-1)^r}{(r+1)^2}$$

and so

$$N\underset{n\to\infty}{-\lim} J_{3n} = -\frac{(-1)^{r+1}}{r+1}x^{r+1}\ln^2 x + \frac{(-1)^{r+1}}{(r+1)^2}x^{r+1}\ln x. \tag{11}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_{x/(x+n)}^{1} u^{-r-2}\ln u \ln(1-u)(1-u)^r \, du \\
&= -\frac{1}{r+1}\int_{x/(x+n)}^{1} \ln u \ln(1-u)(1-u)^r \, d(u^{-r-1}) \\
&= -\frac{(x+n)n^r[\ln x - \ln n - \ln(1+x/n)]\ln(1+x/n)}{(r+1)x^{r+1}} \\
&\quad + \frac{1}{r+1}\int_{x/(x+n)}^{1} u^{-r-2}\ln(1-u)(1-u)^r \, du \\
&\quad - \frac{1}{r+1}\int_{x/(x+n)}^{1} u^{-r-1}\ln u(1-u)^{r-1} \, du \\
&\quad - \frac{r}{r+1}\int_{x/(x+n)}^{1} u^{-r-1}\ln u \ln(1-u)(1-u)^{r-1} \, du.
\end{aligned}$$

The neutrix limit of the first term on the right-hand side of the equation above is equal to

$$\begin{aligned}
&N\underset{n\to\infty}{-\lim} -\frac{(x+n)n^r}{(r+1)x^{r+1}}[\ln x - \ln n - \ln(1+x/n)]\ln(1+x/n) \\
&= \frac{2(-1)^r}{r+1}\left(\frac{1}{r} - \frac{1}{r+1}\right)\ln x + \frac{2(-1)^r}{r+1}\left[\frac{\psi(r-1)}{r} - \frac{\psi(r)}{r+1}\right],
\end{aligned}$$

where

$$\ln^2(1-x) = 2\sum_{n=1}^{\infty}\frac{\psi(n)}{n+1}x^{n+1}.$$

Similarly, if we assume that

$$c_{r-1} = N\underset{n\to\infty}{-\lim}\int_{x/(x+n)}^{1} u^{-r-1}\ln u \ln(1-u)(1-u)^{r-1} \, du$$

exists, it follows that c_r exists and

$$(r+1)c_r + rc_{r-1} = (-1)^r\left[\frac{2\ln x}{r} - \frac{1}{r^2} - \frac{1}{(r+1)^2}\right] + 2(-1)^r\left[\frac{\psi(r-1)}{r} - \frac{\psi(r+1)}{r+1}\right]$$

for $r = 1, 2, \ldots$. Since

$$c_0 = N\underset{n\to\infty}{-\lim}\int_{x/(x+n)}^{1} u^{-2}\ln u \ln(1-u) \, du = \xi(2) - 1 + \frac{1}{2}\ln^2 x$$

ertainly exists, it can be shown by induction that

$$c_r=\frac{2(-1)^r\psi(r)}{r+1}\ln x+\frac{2(-1)^r}{r+1}\sum_{i=1}^{r}\left[\frac{\psi(i-1)}{i}-\frac{\psi(i+1)}{i+1}\right]+\frac{(-1)^r}{r+1}\left[\xi(2)-2+\frac{1}{2}\ln^2 x\right].$$

hus

$$\begin{aligned}N-\lim_{n\to\infty}(-J_{4n})=&\frac{2(-1)^{r+1}\psi(r)}{r+1}x^{r+1}\ln x+\frac{(-1)^{r+1}x^{r+1}}{(r+1)^3}+\frac{2(-1)^{r+1}}{r+1}\\&\times\sum_{i=1}^{r}\left[\frac{\psi(i-1)}{i}-\frac{\psi(i+1)}{i+1}\right]x^{r+1}+\frac{(-1)^{r+1}}{2(r+1)}x^{r+1}\ln^2 x\\&+\frac{(-1)^{r+1}\xi(2)}{r+1}x^{r+1}-\frac{2(-1)^{r+1}}{r+1}x^{r+1}.\end{aligned}\tag{12}$$

inally, it has been shown in [9] that

$$\begin{aligned}N-\lim_{n\to\infty}J_{5n}=&\frac{2(-1)^r\psi(r)}{r+1}x^{r+1}\ln x+\frac{2(-1)^r}{r+1}\sum_{i=1}^{r}\left[\frac{\psi(i-1)}{i}-\frac{\psi(i+1)}{i+1}\right]x^{r+1}\\&+\frac{(-1)^r}{r+1}x^{r+1}\ln^2 x-\frac{2(-1)^r}{r+1}x^{r+1}.\end{aligned}\tag{13}$$

quations (8)–(13) imply that when $x>0$

$$\begin{aligned}N-\lim_{n\to\infty}\int_{-n}^{0}(-y)^r\ln(-y)\ln(x-y)_+\,dy=&\frac{(-1)^{r+1}}{2(r+1)}x_+^{x+1}\ln^2 x_+\\&-\frac{(-1)^{r+1}}{(r+1)^2}x_+^{r+1}\ln x_+ +\frac{(-1)^{r+1}}{(r+1)^3}x_+^{r+1}+\frac{(-1)^{r+1}\xi(2)}{r+1}x_+^{r+1}.\end{aligned}\tag{14}$$

urther, with $a\leqslant x\leqslant b$ and $n>-a$, we have

$$\left|\int_{-n-n^{-n}}^{-n}(-y)^r\ln(-y)\tau_n(y)\ln(x-y)\,dy\right|=O(n^{r-n}\ln^2 n)$$

nd so

$$\lim_{n\to\infty}\int_{-n-n^{-n}}^{-n}(-y)^r\ln(-y)\tau_n(y)\ln(x-y)\,dy=0.\tag{15}$$

t follows from equations (6), (7), (8), (14) and (15) that

$$\begin{aligned}&N-\lim_{n\to\infty}\langle(x_-^r\ln x_-)_n*\ln x_+,\phi(x)\rangle\\&=\left\langle\frac{(-1)^{r+1}x_-^{r+1}\ln^2 x_-}{2(r+1)}+\frac{(-1)^{r+1}x_+^{r+1}\ln^2 x_+}{2(r+1)}-\frac{(-1)^{r+1}}{(r+1)^2}x_+^{r+1}\ln x_+\right.\\&\quad+\frac{\psi(r+1)}{r+1}x_-^{r+1}\ln x_- +\frac{(-1)^{r+1}x_+^{r+1}}{(r+1)^3}+\frac{(-1)^{r+1}\xi(2)}{r+1}x_+^{r+1}\\&\quad\left.-\left[\frac{\psi(r+1)}{r+1}+\xi(2)+\sum_{i=1}^{r}i^{-2}\right]\frac{x_-^{r+1}}{r+1},\ \phi(x)\right\rangle\end{aligned}$$

or $r=1,2,\dots$ and arbitrary ϕ in $\mathscr{D}$.

Theorem 2. *The neutrix convolution product* $(x_-^r \ln x_-) \circledast x_+^{-s}$ *exist and*

$$(x_-^r \ln x_-) \circledast x_+^{-s} = \binom{r}{s-1}\Bigg[\frac{x_-^{r-s+1}\ln^2 x_-}{2} + \frac{(-1)^{r+s}x_+^{r-s+1}\ln^2 x_+}{2}$$

$$-\psi(r-s+1)x_-^{r-s+1}\ln x_- + [\psi(r)-\psi(r-s+1)]x_+^{r-s+1}\mathrm{l}$$

$$+[\chi_s(r)+\Phi_s(r)]x_-^{r-s+1} + (-1)^{r+s}x_+^{r-s+1}\Bigg]$$

for $s = 1, 2, \ldots, r+1$ *and* $r = 1, 2, \ldots$ *and*

$$(x_-^r \ln x_-) \circledast x_+^{-s} = \frac{r!(s-r-2)!}{(s-1)!}\Bigg[x_+^{r-s+1}\ln x_+ + (-1)^{r-s}x_-^{r-s+1}\ln x_-$$

$$+\psi(s-r-2)[x_+^{r-s+1} + (-1)^{r-s}x_-^{r-s+1}] - \psi(r)x_+^{r-s+1}\Bigg]$$

for $s = r+2, r+3, \ldots$ *and* $r = 1, 2, \ldots$ *where*

$$\Phi_s(r) = \sum_{i=1}^{r-1}\frac{\psi(i)}{i+1} - \psi(r-s+1)[\psi(r)-\psi(r-s+2)] + \xi(2)$$

and

$$\chi_s(r) = \sum_{i=1}^{r} i^{-2} - \psi(r)[\psi(r)-\psi(r-s+1)].$$

Proof. If we differentiate equation (4) s times and using equation (3) and the follo two equations

$$(x_+^r \ln^2 x_+)^{(s)} = \frac{r!}{(r-s)!}\Bigg[x_+^{r-s}\ln^2 x_+ + 2[\psi(r)-\psi(r-s)]x_+^{r-s}\ln x_+$$

$$+2\sum_{i=r-s+1}^{r-1}\frac{\psi(i)}{i+1}x_+^{r-s} - 2\psi(r-s)[\psi(r)-\psi(r-s+1)]x_+^{r-}$$

$$(x_+^r \ln x_+)^{(s)} = \frac{r!}{(r-s)!}[x_+^{r-s}\ln x_+ + [\psi(r)-\psi(r-s)]x_+^{r-s}]$$

we get (16). Equation (16) for $s = r+1$ gives

$$x_-^r \ln x_- \circledast x_+^{-r-1} = \frac{1}{2}\ln^2 x_- - \frac{1}{2}\ln^2 x_+ + \psi(r)\ln x_+ + \chi(r)$$

and so by equation (3)

$$-(r+1)(x_-^r \ln x_-) \circledast x_+^{-r-2} = -x_+^{-1}\ln x_+ + x_-^{-1}\ln x_- \psi(r)x_-^{-1}$$

which proves (17) for the case $s = r+2$. Equation (17) now follows easily by induc

References

[1] Antosik P, Mikusinski J and Sikorski R, *Theory of distributions – the sequential approach* (PW Elsevier) (1973)

[2] van der Corput J G, Introduction to the neutrix calculus, *J. Analyse Math.* 7 (1959–1960) 291

[3] Fisher B, Neutrices and the convolution of distributions, *Unv. u Novam Sadu Zb. Rad. Priro Fak. Ser. Mat.* **17** (1987) 119–135

[4] Fisher B and Kaminski A, The neutrix convolution product of distributions $\ln x_-$ and x_+^r, *Proceedings Steklov Inst. Math.* **3** (1995) 233–244
[5] Gel'fand I M and Shilov G E, *Generalized Functions* (Academic Press) (1964) vol. I
[6] Hoskins R and Pinto J S, *Distributions, ultradistributions and other generallized functions* (Ellis Horwood) (1994)
[7] Horvath J, *Topological vector spaces and distributions* (Addison-Wesley, Reading, MA) (1966)
[8] Jones D S, The convolution of generalized functions, *Quart. J. Math.* **24** (1973) 145–163
[9] Özçağ E and Kansu H, A result on neutrix convolution of distributions, *Indian J. Pure Appl. Math.* (to appear)
[10] Vladimirov V S, Equations of mathematical physics, *Nauka* (Moskow) (1968); English translation, (New York: Marcel Dekker) (1971)

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 1, February 1998, pp. 51–53.

A seminorm with square property is automatically submultiplicative

H V DEDANIA

Department of Mathematics, University of Leeds, Leeds LS2 9JT, UK
Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 388 120, India

MS received 11 September 1997

Abstract. The result stated in the title is proved in a linear associative algebra, answering a problem posed in [3].

Keywords. Algebra; seminorm; square property; submultiplicative.

Let $\mathscr{A}$ be a linear associative algebra. A *seminorm on* $\mathscr{A}$ is a function $p:\mathscr{A}\to[0,\infty)$ satisfying the following properties: for $x, y\in\mathscr{A}$ and $\lambda\in\mathbb{C}$,

$$p(x+y)\leqslant p(x)+p(y) \quad \text{and} \quad p(\lambda x)=|\lambda|p(x).$$

A seminorm p has the *square property* [3] if $p(x^2) = p(x)^2$ for all $x\in\mathscr{A}$. It is *submultiplicative* if $p(xy)\leqslant p(x)p(y)$ for all $x, y\in\mathscr{A}$. It is proved in [3] that a seminorm with square property on a commutative algebra is submultiplicative; and it was asked whether the result holds in a noncommutative algebra. By [2], the result holds in Banach algebras. Our main theorem settles the problem completely; it is a uniform seminorm analogue of Sebestyen's Theorem [5, Theorem 2] that every C^*-seminorm is automatically submultiplicative.

First we prove a lemma, which will be the main part of the proof of our theorem.

Lemma. *Let $\mathscr{A}$ be an algebra, and let p be a seminorm on $\mathscr{A}$. Suppose that p has the square property. Then:*

(i) $p(x^n)=p(x)^n \quad (x\in\mathscr{A}, n\in\mathbb{N})$;
(ii) $p(xy+yx)\leqslant 6p(x)p(y) \quad (x, y\in\mathscr{A})$;
(iii) $p(xyx)\leqslant 22p(x)^2p(y) \quad (x, y\in\mathscr{A})$;
(iv) $p(xy-yx)\leqslant \sqrt{300}\,p(x)p(y) \quad (x, y\in\mathscr{A})$.

Proof. (i) Let B_x be the commutative subalgebra of $\mathscr{A}$ generated by x. Then, by [3, Theorem 1], the seminorm p is submultiplicative on B_x. Passing to the completion of (B_x, p) and using the spectral radius formula in Banach algebras we obtain $p(x^n)=p(x)^n$ for all $n\in\mathbb{N}$. This proves (i).

(ii) Let $x_1, y_1\in\mathscr{A}$, and let $\varepsilon>0$. Set $x=x_1/(p(x_1)+\varepsilon)$ and $y=y_1/(p(y_1)+\varepsilon)$. Then $p(x)\leqslant 1$ and $p(y)\leqslant 1$. Now, from the identity $(x+y)^2=x^2+y^2+xy+yx$ and the properties of p we have

$$\begin{aligned} p(xy+yx) &\leqslant p((x+y)^2)+p(x^2)+p(y^2)\\ &= p(x+y)^2+p(x)^2+p(y)^2\\ &\leqslant [p(x)+p(y)]^2+p(x)^2+p(y)^2\\ &\leqslant 6, \end{aligned}$$

and hence that $p(x_1y_1 + y_1x_1) \leqslant 6(p(x_1) + \varepsilon)(p(y_1) + \varepsilon)$. Now (ii) follows, since $x_1, y_1 \in \mathscr{A}$ and $\varepsilon > 0$ are arbitrary.

(iii) Let $x, y \in \mathscr{A}$. Rearranging the identity

$$(x+y)^3 = x^3 + x^2y + xyx + xy^2 + yx^2 + yxy + y^2x + y^3,$$

one gets

$$xyx + yxy = (x+y)^3 - x^3 - y^3 - (x^2y + yx^2) - (xy^2 + y^2x).$$

From (i) and (ii), we have

$$\begin{aligned} p(xyx + yxy) &\leqslant p(x+y)^3 + p(x)^3 + p(y)^3 \\ &\quad + p(x^2y + yx^2) + p(y^2x + xy^2) \\ &\leqslant [p(x) + p(y)]^3 + p(x)^3 + p(y)^3 \\ &\quad + 6p(x)^2p(y) + 6p(x)p(y)^2. \end{aligned}$$

Now suppose that $p(x) \leqslant 1$ and $p(y) \leqslant 1$. Then $p(xyx + yxy) \leqslant 22$. By replacing x by $-x$, we also have $p(xyx - yxy) \leqslant 22$. Since

$$2p(xyx) = p(2(xyx)) \leqslant p(xyx + yxy) + p(xyx - yxy),$$

we have $p(xyx) \leqslant 22$. By a similar argument as in (ii), we obtain $p(xyx) \leqslant 22p(x)^2p(y)$ for all $x, y \in \mathscr{A}$. This proves (iii).

(iv) Note that

$$(xy - yx)^2 + (xy + yx)^2 = 2[(xyx)y + y(xyx)] \; (x, y \in \mathscr{A}).$$

From this identity we have

$$\begin{aligned} p(xy - yx)^2 &= 2p[(xyx)y + y(xyx)] + p(xy + yx)^2 \\ &\leqslant 12p(xyx)p(y) + 36p(x)^2p(y)^2 \quad \text{(by (ii))} \\ &\leqslant 264p(x)^2p(y)^2 + 36p(x)^2p(y)^2 \quad \text{(by (iii))} \\ &= 300p(x)^2p(y)^2. \end{aligned}$$

This proves (iv). □

Theorem. *Let p be a seminorm on $\mathscr{A}$ such that p has the square property. Then p is submultiplicative.*

Proof. Let $x, y \in \mathscr{A}$. Then, by Lemma 1(i) and (ii), we have

$$p(xy) = \frac{1}{2}p(2xy) \leqslant \frac{1}{2}[p(xy + yx) + p(xy - yx)] \leqslant mp(x)p(y),$$

where $m = (\sqrt{300} + 6)/2$.

Set $N_p = \{x \in \mathscr{A} : p(x) = 0\}$. Then N_p is an ideal in $\mathscr{A}$, and so we may assume that p is a norm on $\mathscr{A}$. Define $q(x) = mp(x)$ $(x \in \mathscr{A})$. Then q is a submultiplicative norm satisfying $q(x)^2 = mq(x^2)$ for all $x \in \mathscr{A}$. By the remark following the proof of

[4, Corollary 8, p.77], the algebra $\mathscr{A}$ is commutative. Hence the p is submultiplicative by [3, Theorem 1]. $\square$

Remark. It is proved by Arhippainen [1] that if $\mathscr{A}$ is a commutative locally convex algebra for which the topology is defined by a family of seminorms with square property, then $\mathscr{A}$ is automatically locally m-convex. In fact, by our result, the commutativity of $\mathscr{A}$ follows automatically from the proof.

Acknowledgements

The author is grateful to S J Bhatt for very fruitful discussions and to M H Vasavada for encouragement. The author is also thankful to the National Board for Higher Mathematics, Government of India for a research fellowship.

References

[1] Arhippainen J, On locally convex square algebras. *Funct. Approx. Comment. Math.* **22** (1993) 57–63
[2] Bhatt S J, A seminorm with square property on a Banach algebra is submultiplicative. *Proc. Am. Math. Soc.* **117** (1993) 435–438
[3] Bhatt S J and Karia D J, Uniqueness of the uniform norm with an application to topological algebra. *Proc. Am. Math. Soc.* **116** (1992) 499–503
[4] Bonsall F F and Duncan J, *Complete Normed Algebras* (Springer-Verlag) (1973)
[5] Sebestyen Z, Every C^*-seminorm is automatically submultiplicative. *Period. Math. Hunger.* **10** (1979) 1–8

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 1, February 1998, pp. 55–62.

A class of convolution integral equations involving a generalized polynomial set

S P GOYAL and TARIQ O SALIM

Department of Mathematics, University of Rajasthan, Jaipur 302004, India

MS received 18 March 1997; revised 24 November 1997

Abstract. The aim of this paper is to derive a solution of a certain class of convolution integral equation of Fredholm type whose kernel involves a generalized polynomial set. Our main result is believed to be general and unified in nature. A number of (known or new) results follow as special cases, simply by specializing the coefficients and parameters involved in the generalized polynomial set. For the sake of illustration, some special cases are mentioned briefly.

Keywords. Convolution integral equation; generalized polynomial set; Fox's H-function.

1. Introduction

Raizada [8] introduced and studied a generalized polynomial set which is defined by the following Rodrigues type formula

$$\begin{aligned} & S_n^{\alpha,\beta,\tau}[x; r, c, q, A, B, m, k, l] \\ & \quad = (Ax+B)^{-\alpha}(1-\tau x^r)^{-\beta/\tau}\, T_{k,l}^{m+n}[(Ax+B)^{\alpha+qn}(1-\tau x^r)^{(\beta/\tau)+cn}] \end{aligned} \tag{1}$$

with the differential operator $T_{k,l}$ being defined as

$$T_{k,l} \equiv x^l(k + xD_x), \tag{2}$$

where $D_x \equiv \partial/\partial x$. As $\tau \to 0$, the generalized polynomial set in (1) can be written as

$$\begin{aligned} & S_n^{\alpha,\beta,0}[x; r, q, A, B, m, k, l] \\ & \quad = (Ax+B)^{-\alpha}\exp(\beta x^r)\, T_{k,l}^{m+n}[(Ax+B)^{\alpha+qn}\exp(-\beta x^r)]. \end{aligned} \tag{3}$$

It may be pointed out here that the polynomial set defined by (1) is very general in nature and it unifies and extends a number of classical polynomials introduced and studied by various researchers such as Chartterjea [1, 2], Gould and Hopper [4], Krall and Frink [5], Srivastava and Singhal [11], etc. Some of the special cases of (1) are given by Raizada in tabular form ([8, p. 65]; see also Saigo *et al* [9]).

The main object of this paper is to present an exact solution of the following convolution integral equation of Fredholm type;

$$\int_0^\infty y^{-1} u\left(\frac{x}{y}\right) f(y)\,\mathrm{d}y = g(x) \quad (x > 0), \tag{4}$$

where g is a prescribed function, f is an unknown function to be determined, and the kernel u is given by

$$\begin{aligned} u(x) &= (Ax+B)^{\alpha}(1-\tau x^r)^{\beta/\tau} S_n^{\alpha,\beta,\tau}[x; r, c, q, A, B, m, k, l] \\ &= [x^l(k+xD_x)]^{m+n}\{(Ax+B)^{\alpha+qn}(1-\tau x^r)^{(\beta/\tau)+cn}\}. \end{aligned} \tag{5}$$

Throughout this paper, we assume that $(\alpha + qn) \in N_0$, where N_0 is the set of no negative integers. Note here that if $B = 0$, the restriction $(\alpha + qn) \in N_0$ can be waive

Our method of solution of integral equation (4) with kernel $u(x)$ given by (5) wo depend on the theory of Mellin transform defined by

$$F(s) = M\{f(x); s\} = \int_0^\infty x^{s-1} f(x) \mathrm{d}x,$$

provided that the integral exists.

2. Mellin transform of $u(x)$

In order to solve the integral equation, we shall require the following result.

Lemma. Let $U(s) = M\{u(x); s\}$, *where* $u(x)$ *is defined by* (5), *then*

$$U(s) = \sum_{e=0}^{m+n} \sum_{p=0}^{\alpha+qn} \frac{(-m-n)_e(-\alpha-qn)_p}{e!p!} (-1)^{e+p} k^{m+n-e} B^{\alpha+qn-p} A^p$$

$$l^e \left(-\frac{s+l(m+n)}{l}\right)_e \frac{(-\tau)^{-(s+l(m+n)+p)/r}}{|r|} \Gamma\left(\frac{s+l(m+n)+p}{r}\right)$$

$$\Gamma\left(-\frac{\beta}{\tau} - cn - \frac{s+l(m+n)+p}{r}\right) \left\{\Gamma\left(-\frac{\beta}{\tau} - cn\right)\right\}^{-1},$$

provided that $|\arg \tau| < \pi$, $0 < \mathrm{Re}(s + l(m+n) + p) < r\,\mathrm{Re}(-(\beta/\tau) - cn)$ *when* $r >$ $r\,\mathrm{Re}(-(\beta/\tau) - cn) < \mathrm{Re}(s + l(m+n) + p) < 0$ *when* $r < 0$, $l \neq 0$ *and* $m, n, (\alpha + qn) \in N$

Proof. Making use of binomial expansions for $(Ax + B)^{\alpha+qn}$ and $[x^l(k + x D_x)]^{m+n}$, find that

$$u(x) = \sum_{e=0}^{m+n} \sum_{p=0}^{\alpha+qn} \frac{(-m-n)_e(-\alpha-qn)_p}{e!p!} (-1)^{e+p} k^{m+n-e} B^{\alpha+qn-p} A^p$$

$$x^{l(m+n-e)} (x^{l+1} D_x)^e \{x^p (1 - \tau x^r)^{(\beta/\tau)+cn}\}.$$

Now taking Mellin transform of both sides of eq. (8) and applying the follow known formulas [12, p. 14, eq. (2.2); 3, p. 307, eq. (7)]:

$$M\{(x^{l+1} D_x)^n f(x); s\} = l^n \left(-\frac{s+ln}{l}\right)_n F(s+ln),$$

$$(l \neq 0, n \in N_0)$$

and

$$M\{x^\mu f(x); s\} = F(s+\mu),$$

provided that Mellin transform of (9) and (10) exist, we get

$$U(s) = \sum_{e=0}^{m+n} \sum_{p=0}^{\alpha+qn} \frac{(-m-n)_e(-\alpha-qn)_p}{e!p!} (-1)^{e+p} k^{m+n-e} B^{\alpha+qn-p} A^p$$

$$l^e \left(-\frac{s+l(m+n)}{l}\right)_e M\{x^p (1 - \tau x^r)^{(\beta/\tau)+cn}; s + l(m+n)\}.$$

again, making use of (10) and the following known result [3, p. 311, eq. (30)],

$$M\{(1+ax^r)^{-\nu}; s\} = \frac{1}{|r|} a^{-s/r} B\left(\frac{s}{r}, \nu - \frac{s}{r}\right), \tag{12}$$

where $|\arg a| < \pi$, $0 < \operatorname{Re} s < r \operatorname{Re} \nu$ when $r > 0$; $r \operatorname{Re} \nu < \operatorname{Re} s < 0$ when $r < 0$, and $B(\alpha, \beta)$ denotes the beta function), we arrive at the required result (7).

. Solution of the integral equation (4)

The solution of the convolution integral equation is given in the following theorem.

Theorem. *Let the Mellin transform $F(s)$, $G(s)$ and $U(s) \neq 0$ of the functions $f(x)$, $g(x)$ and $u(x)$ defined by* (5) *exist and be analytic in some infinite strip $\xi < \operatorname{Re} s < \eta$ of the complex s-plane. Also suppose that for a fixed $\gamma \in (\xi, \eta)$, $u^*(x)$ is defined by*

$$u^*(x) = M^{-1}\{U^*(s); x\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} x^{-s} U^*(s)\, ds, \tag{13}$$

where

$$U^*(s) = \left[\mu^L \frac{\Gamma\left(-\frac{s}{\mu}\right)}{\Gamma\left(-L-\frac{s}{\mu}\right)} \sum_{e=0}^{m+n} \sum_{p=0}^{\alpha+qn} \frac{(-m-n)_e(-\alpha-qn)_p}{e!p!} (-1)^p k^{m+n-e} \right.$$

$$B^{\alpha+qn-p} A^p l^e \frac{(-\tau)^{-(s+\mu L+\lambda+l(m+n)+p)/r}}{|r|} \Gamma\left(1 + \frac{s+\mu L+\lambda+l(m+n)}{l}\right)$$

$$\left. \frac{\Gamma\left(\frac{s+\mu L+\lambda+l(m+n)+p}{r}\right)\Gamma\left(-\frac{\beta}{\tau} - cn - \frac{s+\mu L+\lambda+l(m+n)+p}{r}\right)}{\Gamma\left(1+\frac{s+\mu L+\lambda+l(m+n)}{l} - e\right)\Gamma\left(-\frac{\beta}{\tau}-cn\right)} \right]^{-1}, \tag{14}$$

provided that $|\arg \tau| < \pi$, $0 < \operatorname{Re}(s+\mu L+\lambda+l(m+n)+p) < r \operatorname{Re}\left(-\frac{\beta}{\tau} - cn\right)$ *when* $r > 0$; $r \operatorname{Re}\left(-\frac{\beta}{\tau} - cn\right) < \operatorname{Re}(s+\mu L+\lambda+l(m+n)+p) < 0$ *when* $r < 0$, r, l, $\mu \neq 0$ *and m, n, L, $(\alpha+qn) \in N_0$. Then, the integral equation* (4) *has its solution given by*

$$f(x) = x^{-\mu L-\lambda} \int_0^\infty y^{-1} u^*\left(\frac{x}{y}\right) \left(y^{\mu+1} D_y\right)^L \{y^\lambda g(y)\}\, dy, \tag{15}$$

provided that the integral exists.

Proof. Applying the convolution theorem for Mellin transforms [3, p. 308, eq. (14)]
find from (4) that

$$U(s)\, F(s) = G(s),$$

where $U(s)$ is given by (7), and $F(s)$ and $G(s)$ are Mellin transforms of $f(x)$ and
respectively. Replacing s in (16) by $s + \mu L + \lambda$ ($\mu \neq 0$, $L \in N_0$), we obtain

$$F(s + \mu L + \lambda) = U^*(s) \left\{ \mu^L \left(-\frac{s + \mu L}{\mu} \right)_L \right\} G(s + \mu L + \lambda),$$

where $U^*(s)$ is given by (14).

Now using (9) and (10) in (17), we get

$$F(s + \mu L + \lambda) = U^*(s)\, M\{(y^{\mu+1} D_y)^L \{y^\lambda g(y)\}; s\},$$

and making use of (10) and applying the known formula [3, p. 308, eq. (14)] in (18
find that

$$M\,\{x^{\mu L + \lambda} f(x); s\} = M \left\{ \int_0^\infty y^{-1} u^* \left(\frac{x}{y} \right) (y^{\mu+1} D_y)^L \{ y^\lambda g(y) \} \mathrm{d}y; s \right\}.$$

Inverting both sides of (19) by using the Mellin inversion theorem [3, p. 307, eq. (1)]
arrive at the required solution (15).

4. Applications

In view of the relationships of $S_n^{\alpha,\beta,\tau}(x; r, c, q, A, B, m, k, l)$ with various simple class
polynomials listed by Raizada ([8, p. 65]; see also Saigo *et al* [9]), our theorem ca
applied to derive the solution of the convolution integral equations involving clas
Laguerre, Hermite and Bessel polynomials, the Konhauser biorthogonal polynom
and the polynomials $H_n^{(r)}(x, \alpha, \beta)$, $F_n^{(r)}(x, \alpha, q, \beta)$ and $G_n^{(\alpha)}(x, r, \beta, l)$ defined by Gould
Hopper [4], Chatterjea [2] and Srivastava and Singhal [11] respectively.

For example, if we let $\tau \to 0$, then we get the following Corollary.

COROLLARY 1

Under the hypothesis of Theorem, the integral equation

$$\int_0^\infty y^{-1} u_1 \left(\frac{x}{y} \right) f(y)\, \mathrm{d}y = g(x) \quad (x > 0)$$

where the kernel

$$\begin{aligned} u_1(x) &= (Ax + B)^\alpha \exp(-\beta x^r)\, S_n^{\alpha,\beta,0}(x; r, q, A, B, m, k, l) \\ &= [x^l(k + xD_x)]^{m+n} \{(Ax + B)^{\alpha + qn} \exp(-\beta x^r)\}, \end{aligned}$$

has its solution given by

$$f(x) = x^{-\mu L - \lambda} \int_0^\infty y^{-1} u_1{}^* \left(\frac{x}{y} \right) (y^{\mu+1} D_y)^L \{ y^\lambda g(y) \} \mathrm{d}y,$$

provided that the integral exists, and $u_1{}^*(x)$ *is the Mellin inverse transform of*

$$U_1{}^*(s)=\left[\mu^L \frac{\Gamma\left(-\frac{s}{\mu}\right)}{\Gamma\left(-L-\frac{s}{\mu}\right)} \sum_{e=0}^{m+n} \sum_{p=0}^{\alpha+qn} \frac{(-m-n)_e(-\alpha-qn)_p}{e!p!}(-1)^p k^{m+n-e}\right.$$

$$B^{\alpha+qn-p} A^p l^e \frac{\beta^{-(s+\mu L+\lambda+l(m+n)+p)/r}}{|r|}\left\{\Gamma\left(1+\frac{s+\mu L+\lambda+l(m+n)}{l}-e\right)\right\}^{-1}$$

$$\left.\Gamma\left(\frac{s+\mu L+\lambda+l(m+n)+p}{r}\right)\Gamma\left(1+\frac{s+\mu L+\lambda+l(m+n)}{l}\right)\right]^{-1}, \tag{23}$$

provided that $\mathrm{Re}(s+\mu L+\lambda+l(m+n)+p)>0$ *when* $r>0$, $\mathrm{Re}(s+\mu L+\lambda+l(m+n)+p)<0$ *when* $r<0$, $r, l, \mu \neq 0$, and $m, n, L, (\alpha+qn) \in N_0$.

Again, if we put $A=1$, $B=0$ and $k=0$, then we get the following.

COROLLARY 2

Under the hypothesis of Theorem, the integral equation

$$\int_0^\infty y^{-1} u_2\left(\frac{x}{y}\right) f(y)\,\mathrm{d}y=g(x) \quad (x>0) \tag{24}$$

where, the kernel

$$\begin{aligned} u_2(x) &= x^\alpha(1-\tau x^r)^{\beta/\tau} S_n^{\alpha,\beta,\tau}(x; r, c, q, 1, 0, m, 0, l) \\ &= (x^{l+1} D_x)^{m+n}\{x^{\alpha+qn}(1-\tau x^r)^{(\beta/\tau)+cn}\} \end{aligned} \tag{25}$$

has its solution given by

$$f(x)=x^{-\mu L-\lambda} \int_0^\infty y^{-1} u_2{}^*\left(\frac{x}{y}\right)(y^{\mu+1} D_y)^L \{y^\lambda g(y)\}\,\mathrm{d}y, \tag{26}$$

provided that the integral exists, and $u_2{}^*(x)$ *is the Mellin inverse transform of*

$$U_2{}^*(x)=\frac{|r|}{\mu^L l^{m+n}}(-\tau)^{(s+\mu L+\lambda+l(m+n)+\alpha+qn)/r}\,\Gamma\left(-\frac{\beta}{\tau}-cn\right)$$

$$\frac{\Gamma\left(-L-\frac{s}{\mu}\right)\Gamma\left(-(m+n)-\frac{\mu L+\lambda}{l}-\frac{s}{l}\right)}{\Gamma\left(-\frac{s}{\mu}\right)\Gamma\left(-\frac{\mu L+\lambda}{l}-\frac{s}{l}\right)\Gamma\left(\frac{\mu L+\lambda+l(m+n)+\alpha+qn}{r}+\frac{s}{r}\right)}$$

$$\frac{1}{\Gamma\left(-\frac{\beta}{\tau}-cn-\frac{\mu L+\lambda+l(m+n)+\alpha+qn}{r}-\frac{s}{r}\right)}, \tag{27}$$

provided that $|\arg \tau| < \pi$. $0 < \mathrm{Re}(s + \mu L + \lambda + l(m+n) + \alpha + qn) < r\,\mathrm{Re}(-(\beta/\tau) - cn)$ *when* $r > 0$; $r\,\mathrm{Re}(-(\beta/\tau) - cn) < \mathrm{Re}(s + \mu L + \lambda + l(m+n) + \alpha + qn) < 0$ *when* $r < 0$, r, l, $\mu \neq 0$ *and* m, n, $L \in N_0$.

Now, applying the Mellin inversion formula (13), and replacing s by $-s$, we get

$$u_2{}^*(x) = \frac{|r|}{\mu^L l^{m+n}}(-\tau)^{(\mu L + \lambda + l(m+n) + \alpha + qn)/r}\,\Gamma\left(-\frac{\beta}{\tau} - cn\right)$$

$$\frac{1}{2\pi i}\int_{i_\infty}^{\gamma + i\infty}\left(\frac{x}{(-\tau)^{1/r}}\right)^s \Psi(s)\mathrm{d}s, \tag{28}$$

where

$$\Psi(s) = \frac{\Gamma\left(-L + \frac{s}{\mu}\right)\Gamma\left(-(m+n) - \frac{\mu L + \lambda}{l} + \frac{s}{l}\right)}{\Gamma\left(\frac{s}{\mu}\right)\Gamma\left(-\frac{\mu L + \lambda}{l} + \frac{s}{l}\right)\Gamma\left(\frac{\mu L + \lambda + l(m+n) + \alpha + qn}{r} - \frac{s}{r}\right)}$$

$$\frac{1}{\Gamma\left(-\frac{\beta}{\tau} - cn - \frac{\mu L + \lambda + l(m+n) + \alpha + qn}{r} + \frac{s}{r}\right)}, \tag{29}$$

(r, l, $\mu \neq 0$ and m, n, $L \in N_0$).

Under various restrictions on the non-zero constants r, l and μ, the contour integral in (28) can be expressed in terms of Fox's H-function (see for details, Srivastava *et al* [10, ch. 2]). Thus the solution of the integral equation (24) with kernel $u_2(x)$ defined by (25) can be written as

$$f(x) = \frac{r}{\mu^L l^{m+n}}(-\tau)^{(\mu L + \lambda + l(m+n) + \alpha + qn)/r}\,\Gamma\left(-\frac{\beta}{\tau} - cn\right)x^{-\mu L - \lambda}$$

$$\int_0^\infty y^{-1}(y^{\mu+1}D_y)^L\{y^\lambda g(y)\}\,H^{1,1}_{3,3}\left[\frac{x}{y(-\tau)^{1/r}}\middle|\left(1 + l, \frac{1}{\mu}\right),\right.$$

$$\left(-\frac{\mu L + \lambda}{l}, -\frac{1}{l}\right), \left(\frac{\mu L + \lambda + l(m+n) + \alpha + qn}{r}, \frac{1}{r}\right)$$

$$\left(-(m+n) - \frac{\mu L + \lambda}{l}, -\frac{1}{l}\right), \left(1, \frac{1}{\mu}\right),$$

$$\left.\left(1 + \frac{\beta}{\tau} + cn + \frac{\mu L + \lambda + l(m+n) + \alpha + qn}{r}, \frac{1}{r}\right)\right]\mathrm{d}y, \tag{30}$$

for $r > 0$, $l < 0$ and $\mu > 0$, m, n, $L \in N_0$,

$$f(x) = \frac{r}{\mu^L l^{m+n}}(-\tau)^{(\mu L + \lambda + l(m+n) + \alpha + qn)/r}\,\Gamma\left(-\frac{\beta}{\tau} - cn\right)x^{-\mu L - \lambda}$$

$$\int_0^\infty y^{-1}(y^{\mu+1}D_y)^L\{y^\lambda g(y)\}\,H^{2,0}_{3,3}\left[\frac{x}{y(-\tau)^{1/r}}\middle|\begin{matrix}(0, -(1/\mu)),\\ (-L, -(1/\mu)),\end{matrix}\right.$$

$$\left(-\frac{\mu L+\lambda}{l},-\frac{1}{l}\right),\left(\frac{\mu L+\lambda+l(m+n)+\alpha+qn}{r},\frac{1}{r}\right)$$

$$\left(-(m+n)-\frac{\mu L+\lambda}{l},-\frac{1}{l}\right),$$

$$\left(1+\frac{\beta}{\tau}+cn+\frac{\mu L+\lambda+l(m+n)+\alpha+qn}{r},\frac{1}{r}\right)\Bigg]\mathrm{d}y, \tag{31}$$

for $r>0$, $l<0$ and $\mu<0$, $m, n, L\in N_0$, and it is not difficult to find other formulas under various restrictions of r, l and μ.

Also, on setting $l=r=-1$, $q=2$, $m=k=0$, $\alpha=a-2$, $\beta=b$ and $A=1$, $B=0$ as $\tau\to 0$, we get

$$S_n^{(a-2),b,0}[x,-1,2,1,0,0,0,-1]=b^n Y_n(x,a,b), \tag{32}$$

where $Y_n(x,a,b)$ is known as Krall and Frink polynomial [5]. So, we have the following result contained in the following.

COROLLARY 3

Under the hypothesis of Theorem, the integral equation

$$\int_0^\infty y^{-1}u_3(x/y)f(y)\mathrm{d}y=g(x)\quad (x>0) \tag{33}$$

where, the kernel

$$\begin{aligned} u_3(x)&=x^{a-2}\exp(-b/x)\,b^n\,Y_n(x,a,b)\\ &=D_x^n\{x^{a+2n-2}\exp(-b/x)\} \end{aligned} \tag{34}$$

has its solution given by

$$f(x)=x^{-\mu L-\lambda}\int_0^\infty y^{-1}u_3^*(x/y)\,(y^{\mu+1}D_y)^L\{y^\lambda g(y)\}\mathrm{d}y \tag{35}$$

provided that the integral exists, and $u_3^*(x)$ is the Mellin inverse transform of

$$U_3^*(s)=(-1)^n\mu^{-L}b^{-s-\mu L-\lambda a-n+2}\;\frac{\Gamma\left(-L-\frac{s}{\mu}\right)}{\Gamma\left(-\frac{s}{\mu}\right)}$$

$$\frac{\Gamma(s+\mu L+\lambda-n)}{\Gamma(s+\mu L+\lambda)}\quad\frac{1}{\Gamma(-s-\mu L-\lambda-a-n+2)} \tag{36}$$

provided that $\mathrm{Re}(s+\mu L+\lambda+a)<2-n$, $\mathrm{Re}(b)>0$, $n, L\in N_0$ and $\mu\neq 0$.

Putting $L=0$ and $\lambda=1$ in Corollary 3, we get the following.

COROLLARY 4

Under the hypothesis of Theorem, the integral equation (33) *whose kernel is given by* (34) *has its solution given by*

$$f(x)=x^{-1}\int_0^\infty v^*(x/y)\,g(y)\mathrm{d}y \tag{37}$$

provided that the integral exists, and $v^*(x)$ is the Mellin inverse transform of

$$V^*(s) = (-1)^n b^{-s-a-n+1} \frac{\Gamma(s+1-n)}{\Gamma(s+1)\Gamma(-s-a-n+1)} \tag{38}$$

provided that $\mathrm{Re}(s+a) < 1-n$, $\mathrm{Re}(b) > 0$, $\mu \neq 0$ and $n \in N_0$.

If we set $A = 1$, $B = 0$, $k = m = 0$, $\tau \to 0$, and replace α by $(\alpha + ln)$ and q by $-l$ in (5), then we get the solution of the integral involving the kernel

$$u_3(x) = n!\, x^{\alpha + ln} \exp(-\beta x^r)\, G_n^{(\alpha)}(x, r, \beta, l). \tag{39}$$

The above solution has been recently considered by Srivastava [12].

Finally, if we set $A = 1$, $B = 0$, $k = m = q = 0$, $l = \mu = -1$ and $\tau \to 0$ in (5), we get the solution of the integral equation considered by Lala and Shrivastava [6, 7].

Acknowledgements

One of the authors (TOS) wishes to thank Al-Azhar University of Gaza, Palestine, for providing necessary financial assistance for carrying out this investigation. The authors are also grateful to the referee for his valuable suggestions.

References

[1] Chartterjea S K, Some operational formulas connected with a function defined by a generalized Rodrigues' formula, *Acta Math. Acad. Sci. Hungar.* **17** (1996) 379–385

[2] Chartterjea S K, Quelques fonctions génératrics des polynomes d'Hermite du point de vue de algebre de Lie, *C. R. Acad. Sci. Paris, Ser A-B* **268** (1969) 600--602

[3] Erdélyi A, Magnus W, Oberhettinger F and Triconi F G, *Tables of Integral Transforms* (New York: McGraw-Hill) (1954) vol. I

[4] Gould H W and Hopper A T, Operational formulas connected with two generalizations of Hermite polynomials, *Duke Math. J.* **29** (1962) 51–63

[5] Krall H L and Frink O, A new class of orthogonal polynomials; the Bessel polynomials, *Trans. Am. Math. Soc.* **65** (1949) 100–115

[6] Lala A and Shrivastava P N, Inversion of an integral involving a generalized Hermite polynomial, *Indian J. Pure Appl. Math.* **21** (1990) 163–166

[7] Lala A and Shrivastava P N, Inversion of an integral involving a generalized function, *Bull. Calcutta Math. Soc.* **82** (1990) 115–118

[8] Raizada S K, *A study of unified representation of special functions of mathematical physics and their use in statistical and boundary value problems*, Ph. D. thesis (Bundelkhand Univ., Jhansi, India) (1991)

[9] Saigo M, Goyal S P and Saxena S, *A theorem relating a generalized Weyl fractional integral, Laplace and Varma transforms with applications* (communicated for publication)

[10] Srivastava H M, Gupta K C and Goyal S P, *The H-functions of One and Two Variables with Applications* (New Delhi: South Asian Publ.) (1982)

[11] Srivastava H M and Singhal J P, A class of polynomials defined by generalized Rodrigues formula, *Ann. Mat. Pura Appl.* **90** (1971) 75–85

[12] Srivastava R, The inversion of an integral equation involving a general class of polynomials, *J. Math. Anal. Appl.* **186** (1994) 11–20

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 1, February 1998, pp. 63–68.

L^p inequalities for polynomials with restricted zeros

ABDUL AZIZ and W M SHAH

Postgraduate Department of Mathematics and Statistics, University of Kashmir, Hazratbal, Srinagar 190 006, India

MS received 24 December 1996; revised 17 November 1997

Abstract. Let $P(z)$ be a polynomial of degree n which does not vanish in the disk $|z| < k$. It has been proved that for each $p > 0$ and $k \geqslant 1$,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}|P^{(s)}(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant n(n-1)\cdots(n-s+1)B_p \times\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^p d\theta\right\}^{1/p},$$

where $B_p = \left\{\frac{1}{2\pi}\int_0^{2\pi}|k^s + e^{i\alpha}|^p d\alpha\right\}^{-1/p}$ and $P^{(s)}(z)$ is the sth derivative of $P(z)$. This result generalizes well-known inequality due to De Bruijn. As $p \to \infty$, it gives an inequality due to Govil and Rahman which as a special case gives a result conjectured by Erdös and first proved by Lax.

Keywords. Derivative of a polynomial; Zygmund's inequality; L^p norm of a polynomial.

1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ its derivative, then for each $p \geqslant 1$

$$\left\{\int_0^{2\pi}|P'(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant n\left\{\int_0^{2\pi}|P(e^{i\theta})|^p d\theta\right\}^{1/p}. \tag{1}$$

Inequality (1) is due to Zygmund [6] who proved it for all trigonometric polynomials of degree n and not only for those which are of the form $P(e^{i\theta})$. Arestov [1] proved that (1) remains true for $0 < p < 1$ as well.

Inequality (1) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < k$, where $k \geqslant 1$. In the case where $k = 1$, it was found out by De Bruijn [2] that (1) can be replaced by

$$\left\{\int_0^{2\pi}|P'(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant nC_p\left\{\int_0^{2\pi}|P(e^{i\theta})|^p d\theta\right\}^{1/p}, \tag{2}$$

where

$$C_p = \left\{\frac{1}{2\pi}\int_0^{2\pi}|1 + e^{i\alpha}|^p d\alpha\right\}^{-1/p}.$$

For $k \geqslant 1$ Govil and Rahman [4] have shown that if $P(z)$ does not vanish in $|z| < k$, then for every $p \geqslant 1$,

$$\left\{\int_0^{2\pi}|P'(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant nE_p\left\{\int_0^{2\pi}|P(e^{i\theta})|^p d\theta\right\}^{1/p}, \tag{3}$$

where

$$E_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |k + e^{i\alpha}|^p d\alpha\right\}^{-1/p}.$$

Recently, Rahman and Schmeisser [5] have proved that (2) remains true for $0 < p < 1$ also. In case $p = 2$, Dewan and Govil [3] have extended (3) to the sth derivative of $P(z)$ by showing that

$$\int_0^{2\pi} |P^{(s)}(e^{i\theta})|^2 d\theta \leqslant \frac{n^2(n-1)^2\cdots(n-s+1)^2}{1+k^{2s}} \int_0^{2\pi} |P(e^{i\theta})|^2 d\theta, \tag{4}$$

where $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \geqslant 1$.

In this paper we extend inequality (3) to the sth derivative of a polynomial having no zeros in $|z| < k$ where $k \geqslant 1$ and thereby present a generalization of (4). In particular we show that (3) remains true for $0 < p < 1$ as well. More precisely we prove the following theorem.

Theorem. *Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n which does not vanish in $|z| < k$, $k \geqslant 1$. Then for every $p > 0$,*

$$\left\{\int_0^{2\pi} |P^{(s)}(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant n(n-1)\cdots(n-s+1) B_p \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}, \tag{5}$$

where

$$B_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |k^s + e^{i\alpha}|^p d\alpha\right\}^{-1/p}.$$

Remark 1. Inequality (4) of Dewan and Govil [3] is a special case of our theorem, when $p = 2$.

Remark 2. Another special case of our theorem is a result due to Govil and Rahman [5, Theorem 4], which follows from inequality (5) by letting $p \to \infty$ and noting that

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p} = \max_{|z|=1} |P(z)|.$$

2. Lemmas

For $\delta = (\delta_0, \delta_1, \ldots, \delta_n) \in c^{n+1}$ and $P(z) = \sum_{j=0}^n a_j z^j$, we define

$$\Lambda_\delta P(z) = \sum_{j=0}^n \delta_j a_j z^j.$$

The operator Λ_δ is said to be admissible if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in c : |z| \leqslant 1\}$

(ii) $P(z)$ has all its zeros in $\{z \in c : |z| \geqslant 1\}$.

Lemma 1 [1, *Theorem* 4]. *Let $\phi(x) = \psi(\log x)$, where ψ is a convex non-decreasing function on R. Then for all polynomials of degree at most n and each admissible operator Λ,*

$$\int_0^{2\pi} \phi(|\Lambda_\delta P(e^{i\theta})|) d\theta \leqslant \int_0^{2\pi} \phi(c(\delta, n)|P(e^{i\theta})|) d\theta, \tag{6}$$

where $C(\delta, n) = \max\{|\delta_0|, |\delta_n|\}$.

In particular, lemma 1 applies with $\phi: x \mapsto x^p$ for every $p \in (0, \infty)$ and with $\phi: x \mapsto \log x$ as well. Therefore, we have

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |\Lambda_\delta P(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant c(\delta, n)\left\{\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}, \tag{7}$$

where $0 < p < 1$.

We also need the following Lemma.

Lemma 2. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n which does not vanish in* $|z| < k$, $k \geqslant 1$ *and if* $Q(z) = z^n \overline{P(1/\bar{z})}$, *then*

$$k^s |P^{(s)}(z)| \leqslant |Q^{(s)}(z)| \quad \textit{for} \quad |z| = 1, 1 \leqslant s \leqslant n. \tag{8}$$

This lemma is implicit in [4, p. 511]; however for the sake of completeness here we present a simple and an independent proof of this result.

Proof of Lemma 2. By hypothesis all the zeros of the polynomial

$$P(z) = a_n z^n + \cdots + a_s z^s + \cdots + a_1 z + a_0$$

lie in $|z| \geqslant k \geqslant 1$. Therefore, all the zeros of the polynomial

$$F(z) = P(kz) = a_n k^n z^n + \cdots + a_s k^s z^s + \cdots + a_1 kz + a_0 \tag{9}$$

lie in $|z| \geqslant 1$. If we take

$$\begin{aligned} G(z) &= z^n \overline{F(1/\bar{z})} = z^n \overline{P(k/\bar{z})} \\ &= \bar{a}_n k^n + \bar{a}_{n-1} k^{n-1} z + \cdots + \bar{a}_{n-s} k^{n-s} z^s + \cdots + \bar{a}_1 kz^{n-1} + \bar{a}_0 z^n, \end{aligned} \tag{10}$$

then all the zeros of $G(z)$ lie in $|z| \leqslant 1$. Thus the function $f(z) = G(z)/F(z)$ is analytic in $|z| \leqslant 1$ and

$$|f(z)| = |G(z)/F(z)| = |G(z)|/|F(z)| = 1 \quad \textit{for} \quad |z| = 1.$$

By the maximum modulus principle, we have

$$|G(z)/F(z)| = |f(z)| \leqslant 1 \quad \text{for} \quad |z| \leqslant 1. \tag{11}$$

Replacing z by $1/\bar{z}$ in (11), we conclude that

$$|F(z)| \leqslant |G(z)| \quad \text{for} \quad |z| \geqslant 1. \tag{12}$$

Hence for every real or complex number λ with $|\lambda| > 1$, the polynomial $T(z) = F(z) + \lambda G(z)$ has all its zeros in $|z| \leqslant 1$, because if this is not true, then there is a point $z = z_0$ with $|z_0| > 1$ such that $T(z_0) = 0$. This gives

$$0 = T(z_0) = F(z_0) + \lambda G(z_0), \quad \text{where } |z_0| > 1.$$

This implies

$$|F(z_0)| = |\lambda|\,|G(z_0)| > |G(z_0)|, \quad \text{where } |z_0| > 1,$$

which is a contradiction to (12). Thus all the zeros of $F(z) + \lambda G(z)$ lie in $|z| \leqslant 1$ for every complex number λ with $|\lambda| > 1$. By Gauss–Lucas theorem, it follows that all the zeros of $F^{(s)}(z) + \lambda G^{(s)}(z)$ lie in $|z| \leqslant 1$. This implies that

$$|F^{(s)}(z)| \leqslant |G^{(s)}(z)| \quad \text{for } |z| \geqslant 1. \tag{13}$$

Differentiating (10) s times, $1 \leqslant s \leqslant n$, we get

$$G^{(s)}(z) = n(n-1)\cdots(n-s+1)\bar{a}_0 z^{n-s} + \cdots + s!\bar{a}_{n-s}k^{n-s}$$

and by Gauss–Lucas theorem all the zeros of $G^{(s)}(z)$ lie in $|z| \leqslant 1$. If

$$H(z) = z^{n-s}\overline{G^{(s)}(1/\bar{z})} = k^{n-s}a_{n-s}s!z^{n-s} + \cdots + n(n-1)\cdots(n-s+1)a_0,$$

then all the zeros of $H(z)$ lie in $|z| \geqslant 1$ and we see that

$$\begin{aligned} H(z/k) &= a_{n-s}s!z^{n-s} + \cdots + n(n-1)\cdots(n-s+1)a_0 \\ &= z^{n-s}\overline{Q^{(s)}(1/\bar{z})}, \end{aligned}$$

where

$$Q(z) = z^n\overline{P(1/\bar{z})} = \bar{a}_0 z^n + \cdots + \bar{a}_{n-s}z^s + \cdots + \bar{a}_n.$$

Since $F^{(s)}(z) = k^s P^{(s)}(kz)$, from (13) we get

$$k^s|P^{(s)}(kz)| \leqslant |G^{(s)}(z)| = |z^{n-s}\overline{G(1/\bar{z})}| = |H(z)| \quad \text{for } |z| = 1.$$

By the maximum modulus principle,

$$k^s|P^{(s)}(kz)| \leqslant |H(z)| \quad \text{for } |z| \leqslant 1. \tag{14}$$

Taking in particular $z = e^{i\theta}/k$, $0 \leqslant \theta < 2\pi$, so that $|z| = 1/k \leqslant 1$ and from (14) we get

$$k^s|P^{(s)}(e^{i\theta})| \leqslant |H(e^{i\theta}/k)|.$$

Equivalently

$$k^s|P^{(s)}(z)| \leqslant |Q^{(s)}(z)| \quad \text{for } |z| = 1,$$

which is (8) and this completes the proof of the lemma.

3. Proof of the Theorem

First we supose that $k = 1$. Since $P(z)$ is a polynomial of degree $n-s+1$, by repeated application of inequality (1), we have for every $p > 0$,

$$\begin{aligned} \left\{\int_0^{2\pi} |P^{(s)}(e^{i\theta})|^p d\theta\right\}^{1/p} &\leqslant (n-s+1)\left\{\int_0^{2\pi} |P^{(s-1)}(e^{i\theta})|^p d\theta\right\}^{1/p} \\ &\leqslant (n-s+1)(n-s+2)\left\{\int_0^{2\pi} |P^{(s-2)}(e^{i\theta})|^p d\theta\right\}^{1/p} \\ &\cdots \\ &\leqslant (n-s+1)\cdots(n-1)\left\{\int_0^{2\pi} |P'(e^{i\theta})|^p d\theta\right\}^{1/p}. \end{aligned} \tag{15}$$

By hypothesis, $P(z)$ has all its zeros in $|z| \geqslant k = 1$. Using inequality (2) on the right hand side of (15), we get

$$\left\{\int_0^{2\pi} |P^{(s)}(e^{i\theta})|^p d\theta\right\}^{1/p} \leqslant \frac{n(n-1)\cdots(n-s+1)}{\left\{(1/2\pi)\int_0^{2\pi} |1-e^{i\alpha}|^p d\alpha\right\}^{1/p}}\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}$$

for every $p > 0$. This proves (5) for $k = 1$. Henceforth, we suppose $k > 1$. Since $P(z)$ has all its zeros in $|z| \geqslant k > 1$, the polynomial $Q(z) = z^n\overline{P(1/\bar{z})}$ has all its zeros in

$|z| \leqslant 1/k < 1$, which implies by Gauss–Lucas theorem that all the zeros of $Q^{(s)}(z)$ lie in $|z| < 1$. By lemma 2, we have

$$|P^{(s)}(z)| \leqslant \frac{1}{k^s}|Q^{(s)}(z)| < |Q^{(s)}(z)| \quad \text{for } |z| = 1. \tag{16}$$

If $H(z) = z^{n-s}\overline{Q^{(s)}(1/\bar{z})}$, then all the zeros of $H(z)$ lie in $|z| > 1$ and from (16), we have for $|z| = 1$

$$\begin{aligned}|e^{i\alpha}z^s P^{(s)}(z)| &= |P^{(s)}(z)| < |Q^{(s)}(z)| = |z^{n-s}\overline{Q^{(s)}(1/\bar{z})}|\\ &= |H(z)|.\end{aligned}$$

Using Rouche's theorem, it follows that all the zeros of

$$\begin{aligned}\Lambda_\delta P(z) &= H(z) + e^{i\alpha}z^s P^{(s)}(z)\\ &= n(n-1)\cdots(n-s+1)a_0 + (n-1)\cdots(n-s)a_1 z + \cdots + s!a_{n-s}z^{n-s}\\ &\quad + e^{i\alpha}z^s\{n(n-1)\cdots(n-s+1)a_n z^{n-s} + (n-1)\cdots(n-s)a_{n-1}z^{n-s-1}\\ &\quad + \cdots + s!a_s\}\\ &= n(n-1)\cdots(n-s+1)a_0 + (n-1)\cdots(n-s)a_1 z\\ &\quad + \cdots + n(n-1)\cdots(n-s+1)e^{i\alpha}a_n z^n\end{aligned}$$

lie in $|z| > 1$. This shows that the operator Λ_δ is admissible. Hence by (7), we have for every $p > 0$,

$$\int_0^{2\pi} |H(e^{i\theta}) + e^{i\alpha}e^{is\theta}p^{(s)}(e^{i\theta})|^p d\theta \leqslant n(n-1)\cdots(n-s+1)\int_0^{2\pi}|P(e^{i\theta})|^p d\theta. \tag{17}$$

Integrating both sides of (17) with respect to α from 0 to 2π, we get for $p > 0$,

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} |H(e^{i\theta}) + e^{i\alpha}e^{is\theta}p^{(s)}(e^{i\theta})|^p d\alpha d\theta &\leqslant n(n-1)\cdots(n-s+1)\\ &\quad\times 2\pi\int_0^{2\pi}|P(e^{i\theta})|^p d\theta.\end{aligned} \tag{18}$$

Now for every real α and $A \geqslant B \geqslant 1$, it can be easily verified that $|A + e^{i\alpha}| \geqslant |B + e^{i\alpha}|$, which implies for every $p > 0$,

$$\int_0^{2\pi} |A + e^{i\alpha}|^p d\alpha \geqslant \int_0^{2\pi} |B + e^{i\alpha}|^p d\alpha. \tag{19}$$

We take $A = |H(e^{i\theta})/P^{(s)}(e^{i\theta})|$ and $B = k^s > 1$, then by (8), $A \geqslant B \geqslant 1$ and we get with the help of (19) that

$$\begin{aligned}\int_0^{2\pi} |H(e^{i\theta}) + e^{i\alpha}e^{is\theta}p^{(s)}(e^{i\theta})|^p d\alpha &= |P^{(s)}(e^{i\theta})|^p \int_0^{2\pi}\left|\frac{H(e^{i\theta})}{e^{is\theta}P^{(s)}(e^{i\theta})} + e^{i\alpha}\right|^p d\alpha\\ &= |P^{(s)}(e^{i\theta})|^p \int_0^{2\pi}\left|\left|\frac{H(e^{i\theta})}{P^{(s)}(e^{i\theta})}\right| + e^{i\alpha}\right|^p d\alpha\\ &\geqslant |P^{(s)}(e^{i\theta})|^p \int_0^{2\pi} |k^s + e^{i\alpha}|^p d\alpha.\end{aligned}$$

Using this in (18), we conclude for each $p > 0$,

$$\int_0^{2\pi} |P^{(s)}(e^{i\theta})|^p d\theta \leqslant \frac{n(n-1)\cdots(n-s+1)}{(1/2\pi)\int_0^{2\pi} |k^s + e^{i\alpha}|^p d\alpha} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta,$$

from which (5) follows immediately and this completes the proof of the theorem.

References

[1] Arestov V V, On integral inequalities for trigonometric polynomials and their derivatives. *Izv. Aka* *Nauk. SSSR. Ser. Mat.* **45** (1981) 3–22

[2] De Bruijn N G, Inequalities concerning polynomials in the complex domain, *Nederl. Akad. Wetensc* *Proc. Ser.* **A50** (1947) 1265–1272; *Indag. Math.* **9** (1947) 591–598

[3] Dewan K K and Govil N K, Some integral inequalities for polynomials, *Indian J. Pure Appl. Math.* **14** ((1983) 440–443

[4] Govil N K and Rahman Q I, Functions of exponential type not vanishing in a half plane and relate polynomials, *Trans. Am. Math. Soc.* **137** (1969) 501–517

[5] Rahman Q I and Schmeisser G, L^p inequalities for polynomials. J. Approx. Theory **53** (1988) 26–32

[6] Zygmund A, A remark on conjugate series, *Proc. London Math. Soc.* **34** (2) (1932) 392–400

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 1, February 1998, pp. 69–79.

Surface wave propagation in a liquid-saturated porous solid layer lying over a heterogeneous elastic solid half-space

RAJNEESH KUMAR

Department of Mathematics, Kurukshetra University, Kurukshetra 136 119, India

MS received 20 February 1996; revised 22 September 1997

Abstract. Dispersion equation is derived for the propagation of Rayleigh type surface waves in a liquid saturated porous solid layer lying over an inhomogeneous elastic solid half-space. Effect of heterogeneity on the phase velocity is studied by taking different numerical values of heterogeneity factor for particular models. Dispersion curves have been drawn showing the effect of heterogeneity on the phase velocity.

Keywords. Liquid-saturated porous; surface wave; heterogeneous.

1. Introduction

The account of wave propagation in a statistically isotropic medium filled with both non-dissipative and dissipative liquid have been given by Biot [1–3]. Following the theory of Biot, Deresiewicz [4–6] dealt the problem of propagation of surface waves in liquid filled porous solids.

The propagation of Rayleigh waves in a heterogeneous medium was first discussed by Honda [11]. Sezawa [15] studied the general equations of Rayleigh wave propagation in a semi-infinite solid for which the dilatation and the vertical displacement component are zero. Stoneley [16] discussed the propagation of Rayleigh waves in a semi-infinite solid of constant density with its rigidity depending linearly on the depth. Pakeris [14] gave a method to obtain a power series expansion in the wavelength for the Rayleigh wave velocity in a half-space whose density and elastic constants are functions of depth. Dutta [8] studied the propagation of Rayleigh waves in two-layered heterogeneous media. Lal [13] discussed the propagation of Rayleigh waves in an inhomogeneous stratum lying between two semi-infinite homogeneous media.

Practically speaking the earth structure is not homogeneous isotropic throughout. Reasonable grounds are there for the assumption of existence of anisotropy and heterogeneity in the continents, since the material deposited in water are settled in specific directions. We have, therefore, made an attempt to study the problem (two-dimensional) of surface wave propagation in a liquid-saturated porous solid layer lying over a heterogeneous elastic solid half-space.

2. Formulation of the problem

We consider a liquid-saturated porous solid layer of thickness H (medium I) lying over a heterogeneous elastic half-space (medium II). We shall discuss two-dimensional problem in the $x-z$ plane so that the wave front is parallel to $y-z$ plane and displacement components parallel to x and z axis are independent of y. The model assumed is shown in figure 1. Free surface is taken at $z=0$ and interface at $z=H$.

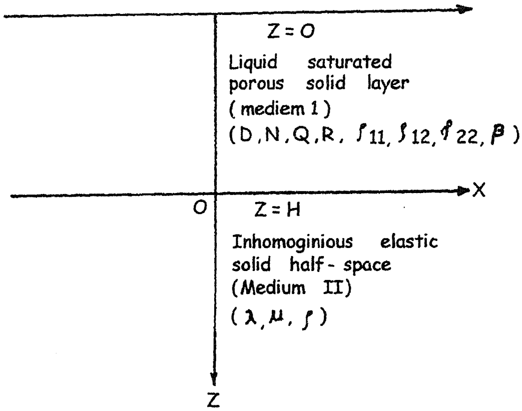

Figure 1. Geometry of the problem.

3. Basic equations and their solutions

In liquid–saturated porous solid (medium I), following Biot [1, 2], equations governing small motions in the absence of dissipation are given by

$$N\nabla^2\mathbf{u} + \text{grad}[(D+N)e + Q\varepsilon] = \frac{\partial^2}{\partial t^2}[\rho_{11}\mathbf{u} + \rho_{12}\mathbf{U}]$$

and

$$\text{grad}[Qe + R\varepsilon] = \frac{\partial^2}{\partial t^2}[\rho_{12}\mathbf{u} + \rho_{22}\mathbf{U}], \tag{1}$$

where **u**, **U** are the displacement vectors of the solid and liquid respectively, and

$$e = \text{div}\,\mathbf{u},$$

$$\varepsilon = \text{div}\,\mathbf{U}.$$

D, N, Q and R are the elastic constants for the solid–liquid aggregate such that D, N correspond to Lame moduli of the material, Q is the measure of coupling between the volume change of the solid and of the liquid and R is the pressure which must be exerted on the liquid to force a given volume of it into the aggregate with the total volume remaining constant.

Stress components in the solid σ_{ij} and the stress in the liquid per unit area σ, are given by

$$\sigma_{ij} = (De + Q\varepsilon)\,\delta_{ij} + 2N\varepsilon_{ij}, \quad \sigma = Qe + R\varepsilon, \tag{2}$$

where δ_{ij} is Kronecker delta, and

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{3}$$

The dynamical coefficients ρ_{11}, ρ_{12} and ρ_{22} in (1) satisfy the relations

$$\rho_{11} + \rho_{12} = (1-\beta_1)\,\rho_s, \quad \rho_{12} + \rho_{22} = \beta_1\,\rho_l\,, \tag{4}$$

where ρ_s, ρ_l denote the mass densities of the solid and of the liquid respectively and β_1 is the porosity of the porous material.

Following Deresiewicz [4], in an unbounded medium assumed to be the non-dissipative liquid-saturated porous solid, there propagate two dilatational waves, 'fast' dilatational or P_f wave, 'slow' dilatational or P_s wave and a shear wave, with the phase velocities λ_1^{-1}, λ_2^{-1} and λ_3^{-1} respectively and these are given by

$$\lambda_1^2 = \frac{B - \sqrt{B^2 - 4AC}}{2A},$$

$$\lambda_2^2 = \frac{B + \sqrt{B^2 - 4AC}}{2A} \quad \text{and} \quad \lambda_3^2 = (C/N)\rho_{22}, \tag{5}$$

where

$$A = (D + 2N)R - Q^2,$$

$$B = \rho_{11}R + \rho_{22}(D + 2N) - 2\rho_{12}Q,$$

$$C = \rho_{11}\rho_{22} - \rho_{12}^2 . \tag{6}$$

Following Biot [1], the following inequalities are satisfied

$$\rho_{11}, \rho_{22} \geq 0, \ \rho_{12} \leqslant 0, \quad A, B, C > 0. \tag{7}$$

For the two-dimensional motion, assuming the wave front parallel to y-axis, the displacements $\mathbf{u} = (u, o, w)$ and $\mathbf{U} = (U, O, W)$ are given by

$$u = \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial x} + \frac{\partial \psi_1}{\partial z},$$

$$w = \frac{\partial \phi_1}{\partial z} + \frac{\partial \phi_2}{\partial z} - \frac{\partial \psi_1}{\partial x},$$

$$U = \mu_1 \frac{\partial \phi_1}{\partial x} + \mu_2 \frac{\partial \phi_2}{\partial x} - \frac{\rho_{12}}{\rho_{22}} \frac{\partial \psi_1}{\partial z},$$

$$W = \mu_1 \frac{\partial \phi_1}{\partial z} + \mu_2 \frac{\partial \phi_2}{\partial z} + \frac{\rho_{12}}{\rho_{22}} \frac{\partial \psi_1}{\partial x}, \tag{8}$$

where ϕ_1, ϕ_2 represent two dilatational wave potentials and ψ_1 represents the shear wave potential in the liquid saturated porous media satisfying the wave equations

$$\nabla^2 \phi_j = \lambda_j^2 \frac{\partial^2 \phi_j}{\partial t^2}, \quad (j = 1, 2)$$

$$\nabla^2 \psi_1 = \lambda_3^2 \frac{\partial^2 \psi_1}{\partial t^2}, \tag{9}$$

and

$$\mu_j = \frac{\rho_{11}R - \rho_{12}Q - A\lambda_j^2}{\rho_{22}Q - \rho_{12}R}, \quad j = 1, 2. \tag{10}$$

Time harmonic solution of the equations (9) can be written as

$$\phi_1 = (A_1 e^{kz\sqrt{1 - c^2\lambda_1^2}} + A_2 e^{-kz\sqrt{1 - c^2\lambda_1^2}}) e^{ik(x - ct)},$$

$$\phi_2 = (A_3 e^{kz\sqrt{1 - c^2\lambda_2^2}} + A_4 e^{-kz\sqrt{1 - c^2\lambda_2^2}}) e^{ik(x - ct)},$$

$$\psi_1 = (B_1 e^{kz\sqrt{1 - c^2\lambda_3^2}} + B_2 e^{-kz\sqrt{1 - c^2\lambda_3^2}}) e^{ik(x - ct)}, \tag{11}$$

where c is the phase velocity and k is the wave number.

The equation of motion for heterogeneous elastic half-space (medium II) is given by Hook [12] and Gupta [10]

$$\gamma\nabla(\mu\nabla\cdot\mathbf{s}) - \nabla x(\mu\nabla x\,\mathbf{s}) + 2\mu'\left[\frac{\partial \mathbf{s}}{\partial z} - i_z \times (\nabla \times \mathbf{s})\right] = \rho\frac{\partial^2 \mathbf{s}}{\partial t^2}, \tag{12}$$

where the displacement vector $\mathbf{s} = \mu\nabla(\mu^{-1}\phi^*) + \mu^{-1}\nabla \times (\mu\psi^*)$,

$$\gamma = (\lambda + 2\mu)/\mu,$$

$$\mu' = \frac{\partial \mu}{\partial z},$$

$\mathbf{i}_z$ = unit vector in the direction of z-axis, and in the elastic half-space medium II, we have assumed the quadratic inhomogeneity varying with the depth, the elastic parameters for which are given by

$$\lambda = \lambda_0(1 + bz)^2,$$
$$\mu = \mu_0(1 + bz)^2,$$
$$\rho = \rho_0, \tag{13}$$

where λ_0, μ_0, ρ_0, and b are arbitrary constants.

For the motion in x–z plane, displacement $\mathbf{s} = (u^*, 0, w^*)$ will be given by

$$u^* = \frac{\partial \phi^*}{\partial x} - \frac{\partial \psi^*}{\partial z} - \frac{2b}{1 + bz}\psi^*,$$

$$w^* = \frac{\partial \phi^*}{\partial z} - \frac{2b}{1 + bz}\phi^* + \frac{\partial \psi^*}{\partial x}. \tag{14}$$

ψ^* is the Y component of $\vec{\psi}^*$.

The wave potentials ϕ^* and ψ^* with harmonic time dependence satisfy the wave equations

$$\nabla^2\phi^* + \frac{\omega^2}{\alpha^2}\phi^* = 0,$$

$$\nabla^2\psi^* + \frac{\omega^2}{\beta^2}\psi^* = 0, \tag{15}$$

where α and β are the longitudinal and transverse velocities in the inhomogeneous medium and are given by

$$\alpha = \alpha_0(1 + bz), \quad \alpha_0 = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}},$$

$$\beta = \beta_0(1 + bz), \quad \beta_0 = \sqrt{\frac{\mu_0}{\rho_0}},$$

$$\omega = kc. \tag{16}$$

Solutions of (15) are given by

$$\phi^* = A e^{-rz} e^{ik(x - ct)},$$

$$\psi^* = B e^{-sz} e^{ik(x - ct)}, \tag{17}$$

here

$$r = k\sqrt{1 - \frac{c^2}{\alpha^2}}, \tag{18}$$

$$s = k\sqrt{1 - \frac{c^2}{\beta^2}}. \tag{19}$$

he expressions for stress components for inhomogeneous elastic half-space are ven by

$$\sigma^*_{xz} = \mu_0\left[2\frac{\partial^2 \phi^*}{\partial x \partial z} - \frac{2b}{1+bz}\frac{\partial \phi^*}{\partial x} + \frac{\partial^2 \psi^*}{\partial x^2} - \frac{2b}{1+bz}\frac{\partial \psi^*}{\partial z} - \frac{\partial^2 \psi^*}{\partial z^2} + \frac{2b^2}{(1+bz)^2}\psi^*\right],$$

$$\sigma^*_{zz} = \mu_0\left[\frac{\partial^2 \phi^*}{\partial x^2} + 3\frac{\partial^2 \phi^*}{\partial z^2} - \frac{6b}{1+bz}\frac{\partial \phi^*}{\partial z} + \frac{6b^2}{(1+bz)^2}\phi^* + 2\frac{\partial^2 \psi^*}{\partial x \partial z} - \frac{2b}{1+bz}\frac{\partial \psi^*}{\partial x}\right], \tag{20}$$

. Boundary conditions

'ollowing Deresiewicz and Skalak [7] boundary conditions will be the vanishing of the :resses at free surface and continuity of stresses, displacements and vanishing of the ormal velocity of the liquid relative to the solid in the porous material at the interfaces. Ve thus have eight conditions given by

$$\sigma_{zz} = \sigma_{zx} = \sigma = 0, \quad \text{at} \quad z = 0,$$

nd at $z = H$,

$$\begin{aligned}(\sigma_{zz} + \sigma)_{\mathrm{I}} &= (\sigma^*_{zz})_{\mathrm{II}},\\ (\sigma_{zx})_{\mathrm{I}} &= (\sigma^*_{zx})_{\mathrm{II}},\\ (u)_{\mathrm{I}} &= (u^*)_{\mathrm{II}},\\ (w)_{\mathrm{I}} &= (w^*)_{\mathrm{II}},\\ (\dot{w} - \dot{W})_{\mathrm{I}} &= 0.\end{aligned} \tag{21}$$

Substituting the values of the potential functions ϕ_1, ϕ_2, ψ_1 from (11) and ϕ^*, ψ^* from (17) in the boundary conditions (21) and making use of (2), (3), (8), (14) and (21), ve get eight linear homogeneous algebraic equations involving the unknowns A_1, A_2, A_3, A_4, B_1, B_2, A and B. The frequency equation is obtained by eliminating the ınknown coefficients and is given by

$$|a_{ij}| = 0, \quad (i, j = 1, 2, \ldots, 8), \tag{22}$$

vhere a_{ij} are the elements of 8×8 matrix $\{a_{ij}\}$ and these are given by

$a_{11} = T^*_1$, $a_{12} = T^*_1$,
$a_{13} = T^*_2$, $a_{14} = T^*_2$,
$a_{15} = -2\xi_3$, $a_{16} = -2\xi_3$,

$$
\begin{aligned}
&a_{17}=0, && a_{18}=0,\\
&a_{21}=2\xi_1, && a_{22}=-2\xi_1,\\
&a_{23}=2\xi_2, && a_{24}=-2\xi_2,\\
&a_{25}=-(1+\xi_3^2), && a_{26}=(1+\xi_3^2),\\
&a_{27}=0=a_{28}, && \\
&a_{31}=(Q+R\mu_1)\frac{c^2}{N}\lambda_1^2, && a_{32}=(Q+R\mu_1)\frac{c^2}{N}\lambda_1^2,\\
&a_{33}=(Q+R\mu_2)\frac{c^2}{N}\lambda_2^2, && a_{34}=(Q+R\mu_2)\frac{c^2}{N}\lambda_2^2,\\
&a_{35}=a_{36}=a_{37}=a_{38}=0, && \\
&a_{41}=T_1e^{kH\xi_1}, && a_{42}=T_1e^{-kH\xi_1},\\
&a_{43}=T_2e^{kH\xi_1}, && a_{44}=T_2e^{-kH\xi_2},\\
&a_{45}=-2\xi_3e^{kH\xi_3}, && a_{46}=-2\xi_3e^{-kH\xi_3},\\
&a_{47}=-S_1e^{-kHr|_{z=H}}, && a_{48}=R_2e^{-kHs|_{z=H}},\\
&a_{51}=2\xi_1e^{kH\xi_1} && a_{52}=-2\xi_1e^{-kH\xi_1},\\
&a_{53}=2\xi_2e^{kH\xi_2} && a_{54}=-2\xi_2e^{-kH\xi_2},\\
&a_{55}=-(1+\xi_3^2)e^{kH_3}, && a_{56}=(1+\xi_3^2)e^{-kH\xi_3},\\
&a_{57}=R_1e^{-kHr|_{z=H}}, && a_{58}=S_2e^{-kHs|_{z=H}},\\
&a_{61}=\xi_1e^{kH\xi_1}, && a_{62}=-\xi_1e^{-kH\xi_1},\\
&a_{63}=\xi_2e^{kH\xi_2}, && a_{64}=-\xi_2e^{-kH\xi_2},\\
&a_{65}=-e^{kH\xi_3}, && a_{66}=e^{-kH\xi_3},\\
&a_{67}=Q_1e^{-kHr|_{z=H}}, && a_{68}=-e^{-kHs|_{z=H}},\\
&a_{71}=e^{kH\xi_1}, && a_{72}=e^{-kH\xi_1},\\
&a_{73}=e^{kH\xi_2}, && a_{74}=e^{-kH\xi_2},\\
&a_{75}=-\xi_3e^{kH\xi_3}, && a_{76}=-\xi_3e^{-kH\xi_3},\\
&a_{77}=-e^{-kHr|_{z=H}}, && a_{78}=Q_2e^{-kHs|_{z=H}},\\
&a_{81}=(1-\mu_1)\xi_1e^{kH\xi_1}, && a_{82}=-(1-\mu_1)\xi_1e^{-kH\xi_1},\\
&a_{83}=(1-\mu_2)\xi_2e^{kH\xi_2}, && a_{84}=-(1-\mu_2)\xi_2e^{-kH\xi_2},\\
&a_{85}=-(1-\alpha_0')e^{kH\xi_3}, && a_{86}=(1-\alpha_0')e^{-kH\xi_3},\\
&a_{87}=0=a_{88}, &&
\end{aligned}
\tag{2.}
$$

where

$$T_j^*=2-c^2\lambda_j\frac{(P+Q\mu_j)}{N},\quad j=1,2,$$

$$T_j=2-\frac{c^2}{N}\lambda_j^2\,(P+Q+(Q+R)\mu_j],\quad j=1,2,$$

$$Q_1=\frac{1}{kH}\left[kHr|_{z=H}+kH^2r_z|_{z=H}+\frac{2bH}{1+bH}\right],$$

$$Q_2=\frac{1}{kH}\left[kHs|_{z=H}+kH^2s_z|_{z=H}-\frac{2bH}{1+bH}\right],$$

$$R_1=\frac{2\mu^*}{N}\cdot\frac{1}{kH}\left[kHr|_{z=H}+kH^2r_z|_{z=H}+\frac{bH}{1+bH}\right],$$

$$R_2=\frac{2\mu^*}{N}\cdot\frac{1}{kH}\left[kHs|_{z=H}+kH^2s_z|_{z=H}+\frac{bH}{1+bH}\right],$$

$$S_1 = \frac{\mu^*}{N}\cdot\frac{1}{k^2H^2}\left[\frac{6bH}{1+bH}\left\{kHr|_{z=H} + kH^2r_z|_{z=H} + \frac{bH}{1+bH}\right\}\right.$$
$$+3\left\{kHr|_{z=H} + kH^2r_z|_{z=H}\right\}^2$$
$$\left. - 6kH^2r_z|_{z=H} - 3kH^3r_{zz}|_{z=H} - k^2H^2\right],$$

$$S_2 = \frac{\mu^*}{N}\cdot\frac{1}{k^2H^2}\left[\frac{2bH}{1+bH}\left(kHs|_{z=H} + kH^2s_z|_{z=H} + \frac{bH}{1+bH}\right)\right.$$
$$\left. - \left(kHs|_{z=H} + kH^2s_z|_{z=H}\right)^2 + 2kH^2s_z|_{z=H} + kH^3s_{zz}|_{z=H} - k^2H^2\right]$$

$$kHr|_{z=H} = \frac{kH}{1+bH}\left[(1+bH)^2 - \frac{c^2}{\alpha_0^2}\right]^{1/2},$$

$$kH^2r_z|_{z=H} = kbH^2\frac{c^2}{\alpha_0^2}\frac{1}{(1+bH)^2[(1+bH)^2-(c^2/\alpha_0^2)]^{1/2}},$$

$$kH^3r_{zz}|_{z=H} = -kH^3b^2\frac{c^2}{\alpha_0^2}\left[\frac{(1+bH)^2 + 2\{(1+bH)^2-(c^2/\alpha_0^2)\}}{(1+bH)^3\{(1+bH)^2-(c^2/\alpha_0^2)\}^{3/2}}\right],$$

$$kHs|_{z=H} = \frac{kH}{1+bH}\left[(1+bH)^2 - \frac{c^2}{\beta_0^2}\right]^{1/2},$$

$$kH^2s_z|_{z=H} = kbH^2\frac{c^2}{\beta_0^2}\frac{1}{(1+bH)^2}\frac{1}{[(1+bH)^2-(c^2/\beta_0^2)]^{1/2}},$$

$$kH^3s_{zz}|_{z=H} = -kH^3b^2\frac{c^2}{\beta_0^2}\left[\frac{(1+bH)^2 + 2\{(1+bH)^2-(c^2/\beta_0^2)\}}{(1+bH)^3\{(1+bH)^2-(c^2/\beta_0^2)\}^{3/2}}\right],$$

$$\xi_j = \sqrt{1-c^2\lambda_j^2}, \quad (j=1,2,3)$$

$$P = D + 2N. \tag{24}$$

Equation (23) is the required frequency equation relating phase velocity c to the wave length $2\pi/k$. The wave length is a multivalued function of phase velocity, each value corresponds to a different mode of propagation and also indicates the dispersive nature of the existing wave. Such a surface wave can exist if and only if equation (23) has a real solution, i.e., there should be at least one value of c for which r and s are real and positive.

We notice that when ξ_1, ξ_2 and ξ_3 are real, all the entries a_{ij} $(i, j = 1, \ldots)$ given by equation (23) are real otherwise:

(i) When ξ_1 is not real, i.e., $\xi_1 = i\xi_1'$ (say) $\xi_1' \in R^*$, then in order to get all the entries involving ξ_1, i.e., first and second column, to be real it is required to transform these column by column operations to:

$$a_{11} = T_1, \qquad a_{12} = 0,$$
$$a_{21} = 2\xi_1, \qquad a_{22} = 0,$$
$$a_{31} = (Q + R\mu_1)\frac{c^2}{N}\lambda_1^2, \qquad a_{32} = 0,$$

$$a_{41} = T_1 \cosh kH\xi'_1, \qquad a_{42} = T_1 \sinh kH\xi'_1,$$
$$a_{51} = -2\xi'_1 \sinh kH\xi'_1, \qquad a_{52} = 2'_1 \cosh kH\xi'_1,$$
$$a_{61} = -\xi'_1 \sinh kH\xi'_1, \qquad a_{62} = \xi'_1 \cosh kH\xi'_1,$$
$$a_{71} = \cosh kH\xi'_1, \qquad a_{72} = \sinh kH\xi'_1,$$
$$a_{81} = -(1-\mu_1)\xi'_1 \sinh kH\xi'_1, \qquad a_{82} = (1-\mu_1)\xi'_1 \cosh kH\xi'_1.$$

(ii) when ξ_2 is not real, i.e., $\xi_2 = i\xi'_2$ (say) $\xi'_2 \in R$, then in order to get all the entries involving ξ_2, i.e., third and fourth column, to be real, it is required to transform these column by column operations to

$$a_{13} = -T_2, \qquad a_{14} = 0,$$
$$a_{23} = 0, \qquad a_{24} = 2\xi'_2,$$
$$a_{33} = \frac{Q}{N}(Q + R\mu_2)(\xi'_2 + 1), \qquad a_{34} = 0,$$
$$a_{43} = T_2 \cos kH\xi'_2, \qquad a_{44} = T_2 \sin kH\xi'_2,$$
$$a_{53} = -2\xi'_2 \sin kH\xi'_2, \qquad a_{54} = 2\xi'_2 \cos kH\xi'_2,$$
$$a_{63} = -\xi'_2 \sin kH\xi'_2, \qquad a_{64} = \xi'_2 \cos kH\xi'_2,$$
$$a_{73} = \cos kH\xi'_2, \qquad a_{74} = \sin kH\xi'_2,$$
$$a_{83} = -(1-\mu_2)\xi'_2 \sin kH\xi'_2, \qquad a_{84} = (1-\mu_2)\xi'_2 \cos kH\xi'_2.$$

(iii) when ξ_3 is not real, i.e., $\xi_3 = i\xi'_3$ (say), $\xi'_3 \in R$, then in order to get all the entries involving ξ_3, i.e., fifth and sixth column, to be real, it is required to transform these column by column operations to

$$a_{15} = 2\xi'_3, \qquad a_{16} = 0,$$
$$a_{25} = 0, \qquad a_{26} = -(1-\xi'^2_3),$$
$$a_{35} = 0, \qquad a_{36} = 0,$$
$$a_{45} = -2\xi'_3 \cosh kH\xi'_3, \qquad a_{46} = 2\xi'_3 \sin kH\xi'_3,$$
$$a_{55} = -(1-\xi'^2_3) \sin kH\xi'_3, \qquad a_{56} = -(1-\xi'^2_3) \cosh kH\xi'_3,$$
$$a_{65} = -\sin kH\xi'_3, \qquad a_{66} = -\cos kH\xi'_3,$$
$$a_{75} = -\xi'_3 \cos kH\xi'_3, \qquad a_{76} = \xi'_3 \sin kH\xi'_3,$$
$$a_{85} = -(1-\alpha'_0) \sin kH\xi'_3, \qquad a_{86} = \xi'_3 \sin kH\xi'_3.$$

Special case

If we put the inhomogeneity factor $b = 0$ in equation (23), we get the frequency equation for surface wave propagation in a homogeneous liquid-saturated porous solid layer over a homogeneous isotropic half-space.

5. Numerical results and discussion

Since large number of parameters appear in the final expressions (23), therefore, in order to discuss the possibility of propagation of surface waves along x-direction, particular models are considered. Actually, we consider two models. In the first model, medium I is taken to be water-saturated sandstone layer. In the second model, medium I is taken to be a layer of kerosene saturated sandstone. For both the models, we calculate the ratios of the phase velocity to the velocity of slow dilatational wave in water saturated sandstone and kerosene saturated sandstone (c/α_2) for the given values of dimensionless number kH and bH.

Keeping in view the experimental results given by Yew and Jogi [17] which differ slightly from the experimental results given by Fatt [9] for (i) water staurated sandstone and (ii) kerosene saturated sandstone. We choose the following values of the relevant parameters:

(i) For water saturated sandstone (medium I):

$P = 2.15 \times 10^{11}\,\text{dyn}\,\text{cm}^{-2}$,
$Q = 0.013 \times 10^{11}\,\text{dyn}\,\text{cm}^{-2}$,
$R = 0.0637 \times 10^{11}\,\text{dyn}\,\text{cm}^{-2}$,
$N = 0.922 \times 10^{11}\,\text{dyn}\,\text{cm}^{-2}$,
$\rho_{11} = 1.9032\,\text{g}\,\text{cm}^{-3}$,
$\rho_{12} = 0.0\,\text{g}\,\text{cm}^{-3}$,
$\rho_{22} = 0.268\,\text{g}\,\text{cm}^{-3}$,
$\beta = 0.268$.

(ii) For kerosene saturated sandstone (medium I):

$P = 0.99663 \times 10^{11}\,\text{dyn}\,\text{cm}^{-2}$,
$Q = 0.07435 \times 10^{11}\,\text{dyn}\,\text{cm}^{-2}$,
$R = 0.03262 \times 10^{11}\,\text{dyn}\,\text{cm}^{-2}$,
$N = 0.2765 \times 10^{11}\,\text{dyn}\,\text{cm}^{-2}$,
$\rho_{11} = 1.926137\,\text{g}\,\text{cm}^{-3}$,
$\rho_{12} = -0.002137\,\text{g}\,\text{cm}^{-3}$,
$\rho_{22} = 0.215337\,\text{g}\,\text{cm}^{-3}$,
$\beta = 0.26$.

For the inhomogeneous elastic half-space (medium II), folloing values of the relevant parameters are chosen:

$$\mu_0 = 3.3 \times 10^{11}\,\text{dyne}\,\text{cm}^{-2},$$

$$\rho_0 = 2.6\,\text{gm}\,\text{cm}^{-3}.$$

It is noticed from figure 2 that for water saturated sandstone, velocity ratio (c/α_2) remains the same as kH varies between $0.0 \leqslant kH \leqslant 1.4$ for $bH = 0.0$, 0.5 and 2.5. For $bH = 0.0$, it increases very rapidly and decreases instantaneously and goes on decreasing as kH increases from 1.4 to 4.9 whereas velocity ratio decreases for the range $1.4 < kH \leqslant 4.9$ for $bH = 0.5$ and 2.5. For the range $4.9 < kH \leqslant 5.2$, velocity ratio remains the same for $bH = 0.0$, 0.5 and 2.5. For $bH = 0.0$, 0.5 and 2.5, velocity ratio increases very rapidly and then decreases gradually for the range $5.2 < kH \leqslant 14.3$. Again velocity ratio increases instantaneously and then decreases with the further increases of kH for $bH = 0.0$, 0.5 and 2.5.

It is clear from figure 3 that for kerosene-saturated sandstone the value of velocity ratio c/α_2 remains constant and same as $0.0 \leqslant kH \leqslant 2.4$ for $bH = 0.0$, 0.5 and 2.5. The velocity ratio increases as inhomogeneity factor bH increases for $2.4 < kH \leqslant 4.9$ and for the range $4.9 < kH \leqslant 6.2$ velocity ratio c/α_2 remains same. The velocity ratio increases rapidly and decreases gradually as kH varies between $6.2 < kH \leqslant 15.0$ for $bH = 0.0$; $6.2 < kH \leqslant 12.1$ for $bH = 0.5$ and $6.2 < kH \leqslant 11.8$ for $bH = 2.5$. With the further increases of kH, velocity ratio decreases rapidly and then remains constant for $bH = 0.0$, 0.5 and 2.5 respectively.

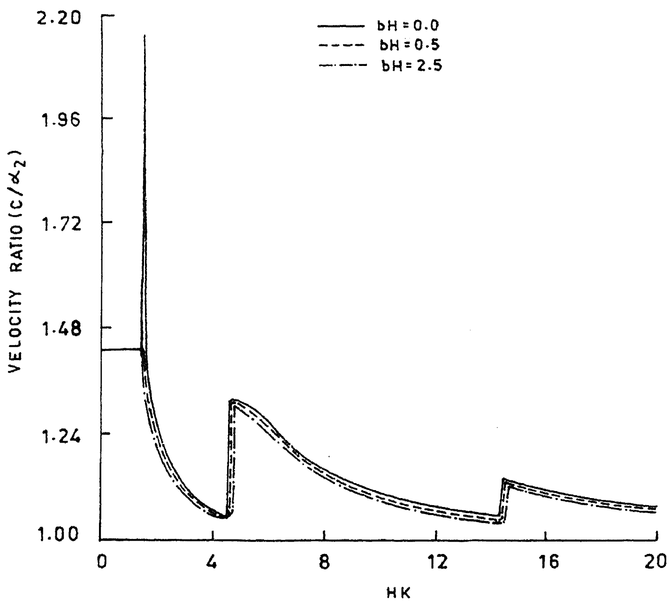

Figure 2. Dispersion curves for phase velocity.

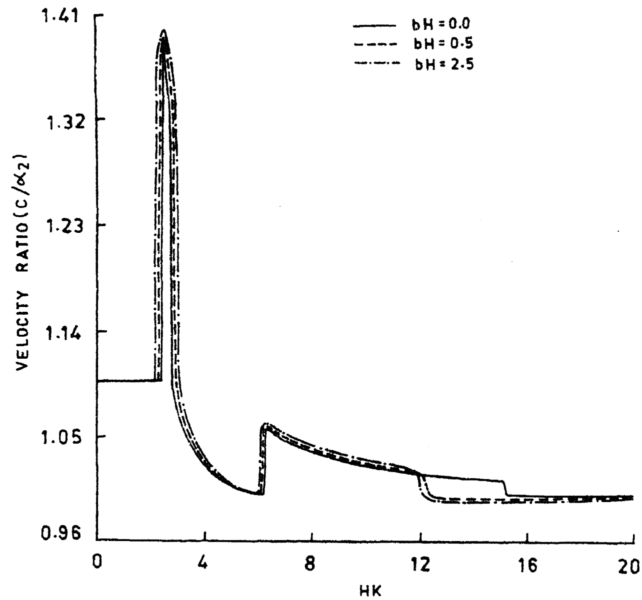

Figure 3. Dispersion curves for phase velocity.

It is observed from figures 2 and 3 that as inhomogeneity factor increases, velocity ratio decreases for water-saturated sandstone model and as inhomogeneity factor increases velocity ratio c/α_2 increases for kerosene saturated sandstone model.

Comparison of figures 2 and 3 show the effect of inhomogeneity in the two proposed models.

Acknowledgement

The author is thankful to the referee for his valuable suggestions.

References

[1] Biot M A, Theory of propagation of elastic waves in a fluid-saturated porous solid. *J. Acoust. Soc. Am.* **28** (1956a) 168–191

[2] Biot M A, Propagation of elastic waves in liquid filled porous solid, *J. Appl. Phys.* **27** (1956b) 459–467

[3] Biot M A, Generalized theory of acoustic propagation in porous dissipative media, *J. Acoust. Soc. Am.* **34** (1962) 1256–1264

[4] Deresiewicz H, The effect of boundaries on wave propagation in a liquid-filled porous solid–I. *Bull. Seis. Soc. Am.* **50** (1960a) 599–607

[5] Deresiewicz H, The effect of boundaries on wave propagation in a liquid filled porous solid–II, *Bull. Seis. Soc. Am.* **5** (1960b) 51–59

[6] Deresiewicz H, The effect of boundaries on wave propagation in a liquid-filled porous solid, *Bull. Seis. Soc. Am.* **52** (1962) 627–638

[7] Deresiewicz H and Skalak R, On uniqueness in dynamic poroelasticity, *Bull. Seis. Soc. Am.* **53** (1963), 783–789

[8] Dutta S, Rayleigh waves in a two-layered heterogeneous medium, *Bull. Seis. Soc. Am.* **53** (1963) 517

[9] Fatt I, The Biot–Willis elastic coefficients for sandstone, *Trans. ASME Ser. E. J. Appl. Mech.* **26** 1(1959) 296–297

[10] Gupta R N, Reflection of elastic waves from a linear transition layer, *Bull. Seis. Soc. Am.* **50** (1966b) 511–526

[11] Honda H, *Kisyo-Syusi* **9** (1931) 237 (in Japanese)

[12] Hook J F, Separation of the vector wave equation of elasticity for inhomogeneous media, *J. Acoust. Soc. Am.* **33** (1961) 302

[13] Lal T, Rayleigh waves in an inhomogeneous interstretum, *Bull. National Geophys. Res. Inst.* **6** (1968) 113–123

[14] Pakeris C L, Propagation of Rayleigh waves in heterogeneous media. *J. Appl. Phys.* **6** (1935) 133–138

[15] Sezawa K, A kind of waves transmitted over a semi-inifinite solid body of varying elasticity. *Bull. Earthq. Res. Inst. (Tokyo)* **9** (1931) 310–315

[16] Stoneley R, The transmission of Rayleigh waves in a heterogeneous medium, *Mon. Not. R. Astr. Soc. Geophys. Suppl.* **3** (1934) 222–232

[17] Yew C H and Jogi P N, Study of wave motions in fluid saturated porous rocks, *J. Acoust. Soc. Am.* **60** (1976) 2–8

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 1, February 1998, pp. 81–94.

Scattering of Love waves due to the presence of a rigid barrier of finite depth in the crustal layer of the earth

P S DESHWAL and S MUDGAL

Department of Mathematics, Maharshi Dayanand University, Rohtak 124 001, India

MS received 29 May 1997; revised 6 October 1997

Abstract. The problem of scattering of Love waves due to the presence of a rigid barrier of finite depth in the crustal layer of the earth is studied in the present paper. The barrier is in the slightly dissipative surface layer and the surface of the layer is a free surface. The Wiener–Hopf technique is the method of solution. Evaluation of the integrals along appropriate contours in the complex plane yields the reflected, transmitted and the scattered waves. The scattered waves behave as a decaying cylindrical wave at distant points. Numerical computations for the amplitude of the scattered waves have been made versus the wave number. The amplitude falls off rapidly as the wave number increases very slowly.

Keywords. Love waves; scattering; surface layer; seismic waves.

1. Introduction

Seismic waves appear on the surface of the earth during an earthquake and let loose their energies around the inhomogeneities and irregularities on the surface of the earth. Love wave is partly responsible for the destruction of buildings and loss of human lives. Scattering of Love waves due to surface defects results in large amplification of waves during earthquakes. The problem of scattering of Love waves due to the presence of a rigid barrier requires investigation.

The paper has its application to scattering of seismic waves due to hard materials like rocks in the surface of the crust. Our problem is based on a paper by Sato [9] who has studied the problem of reflection and transmission of Love waves at a vertical discontinuity in a surface layer. Kazi [7] used a variational method due to Schwinger and Levine to study Love wave propagation across a continental margin. Both of them have taken surface discontinuity in the form of a semi-infinite rectangular strip which simplifies mathematical calculations. Deshwal [2] has discussed the problem of scattering of Love waves due to the presence of an infinite rigid strip in a surface layer. Deshwal [3] has also studied the problem of scattering of Love waves due to a surface impedance in a surface layer. The scattered wave has a logarithmic singularity at the tip of the scatterer and behaves as a decaying cylindrical wave at distant points. Deshwal [4] has further studied Love wave propagation in the case of a rectangular discontinuity in a surface layer. The reflection and transmission of a Love wave due to a line and a semi-infinite rectangular strip are derived as a special case. All these authors except Kazi [7] have used Fourier transformation and Wiener–Hopf technique in their research work. Asghar and Zaman [1] have solved the problem of diffraction of Love waves normally incident on two parallel perfectly weak (crack) half-planes lying in a surface layer and parallel to the interface between the layer and half-space. The problem is formulated in terms of the two Wiener–Hopf equations and is solved by the technique introduced by Jones [6]. Zaman and Asghar [11] have also solved the

problem of dispersion of Love waves in an inhomogeneous layer due to a source and used Green function method. Xiao [10] has discussed the attenuation dispersion of Love waves in an elastic two layered half-space. The attenuation of dispersive Love waves in an elastic two layered half-space and in a simple symmetric homogeneous three-layered model has been investigated by introducing the complex propagation functions into the known explicit dispersion relation.

We discuss the propagation of Love waves in the presence of a rigid barrier $-H \leqslant z \leqslant -h$, $x=0$, in a surface layer $-H \leqslant z \leqslant 0$, $-\infty < x < \infty$, superposed on a solid half-space $z \geqslant 0$, $-\infty < x < \infty$. The method of solution is the Fourier transformation of basic equations and finding unknown functions by the Wiener–Hopf technique [8].

2. Formulation of the problem

The problem is two-dimensional, the zx-plane is the vertical plane and Love waves are propagated parallel to the x-axis. A solid layer of thickness H lies over a solid half-space. The media are homogeneous, isotropic and slightly dissipative. The velocities and rigidities of shear waves in the half-space and the layers are V_1, μ_1 and V_2, μ_2 respectively as shown in figure 1. Let the incident wave be [9]

$$v_{0,1} = A \cos \beta_{2N} H \exp(-\beta_{1N} z - ik_{1N} x), \quad z \geqslant 0, \tag{1}$$

$$v_{0,2} = A \cos \beta_{2N}(z+H) \exp(-ik_{1N} x), \quad -H \leqslant z \leqslant 0, \tag{2}$$

where

$$\beta_{1N} = (k_{1N}^2 - k_1^2)^{1/2}, \quad \beta_{2N} = (k_2^2 - k_{1N}^2)^{1/2}, \quad |k_1| < |k_2|, \tag{3}$$

and k_{1N} is a root of the equation

$$\tan \beta_{2N} H = \gamma \beta_{1N} / \beta_{2N}, \quad \gamma = \mu_1/\mu_2. \tag{4}$$

The wave equation in two dimensions is

$$(\nabla^2 + k_j^2) v_j = 0, \quad j = 1,2, \tag{5}$$

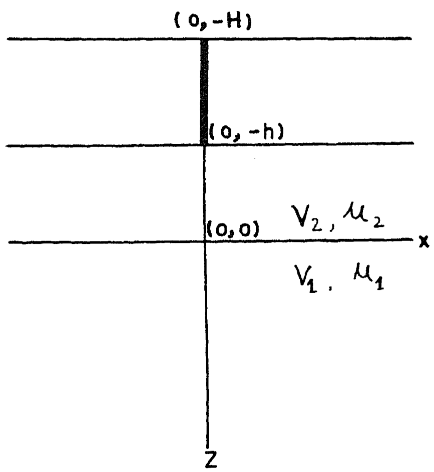

Figure 1. The geometry of the model.

here

$$k_j = \left(\frac{w^2 + i\varepsilon w}{V_j^2}\right)^{1/2} = k_j' + ik_j''. \tag{6}$$

$V_j\,(j=1,2)$ are velocities of shear waves for two media. $\varepsilon > 0$ is a damping constant uch that the medium is slightly dissipative. The displacements $v_j\,(j=1,2)$ have a time ictor $\exp(-iwt)$. k_j are complex with their imaginary parts being small and positive.

. Boundary conditions

et the total displacements in the media be given by

$$v = v_{0,1} + v_1, \quad z \geqslant 0, \quad -\infty < x < \infty \tag{7}$$

$$= v_{0,2} + v_2, \quad -H \leqslant z \leqslant 0, \quad -\infty < x < \infty. \tag{8}$$

he conditions on the boundaries are

$$v_1 = v_2, \quad \frac{\gamma \partial v_1}{\partial z} = \frac{\partial v_2}{\partial z}, \quad z = 0, \quad -\infty < x < \infty, \tag{9}$$

$$v_{0,2} + v_2 = 0, \quad -H \leqslant z \leqslant -h, \quad x = 0, \tag{10}$$

$$\frac{\partial}{\partial z}(v_{0,2} + v_2) = 0, \quad x \geqslant 0, \quad x \leqslant 0, \quad z = -H. \tag{11}$$

he condition (10) implies that there is no displacement across the rigid barrier. This an be simplified to

$$v_2 = -A\cos\beta_{2N}(z+H)\exp(-ik_{1N}x), \quad -H \leqslant z \leqslant -h, \quad x = 0. \tag{12}$$

et us define the Fourier transforms

$$\begin{aligned}\bar{v}_j(p,z) &= \int_{-\infty}^{\infty} v_j(x,z)\exp(ipx)\mathrm{d}x, \quad p = \alpha + i\beta \\ &= \int_{-\infty}^{0} v_j(x,z)\exp(ipx)\mathrm{d}x + \int_{0}^{\infty} v_j(x,z)\exp(ipx)\mathrm{d}x \\ &= \bar{v}_{j-}(p,z) + \bar{v}_{j+}(p,z).\end{aligned} \tag{13}$$

f for given z,

$$|v_j(x,z)| \sim M\exp(-d|x|), \quad \text{as} \quad |x| \to \infty, M, d > 0, \tag{14}$$

ıen $\bar{v}_{j+}(p,z)$ is analytic in $\beta > -d$ and $\bar{v}_{j-}(p,z)$ in $\beta < d$. $\bar{v}_j$ and its derivatives with espect to z are therefore analytic in the strip $-d < \beta < d$ of the complex p-plane.

. Solution of the problem

aking the Fourier transform of (5) results in

$$\frac{\mathrm{d}^2\bar{v}_j(p,z)}{\mathrm{d}z^2} - \beta_j^2\bar{v}_j(p,z) = 0, \quad \beta_j = \pm(p^2 - k_j^2)^{1/2}. \tag{15}$$

The sign for β_j is to be chosen such that its real part is always positive for all p. The solution of (15) is

$$\bar{v}_1(p,z) = B(p)\exp(-\beta_1 z), \quad z \geqslant 0, \tag{16}$$

$$\bar{v}_2(p,z) = C(p)\exp(-\beta_2 z) + D(p)\exp(\beta_2 z), \quad -H \leqslant z \leqslant 0. \tag{17}$$

Using the boundary condition (9), we find

$$\bar{v}_2(p,z) = B(p)[\beta_2 \cosh\beta_2 z - \gamma\beta_1 \sinh\beta_2 z]/\beta_2. \tag{18}$$

Differentiating (18) with respect to z, putting $z = -h$ in the resulting equation, denoting $\bar{v}_j(p,-h)$ and $\bar{v}'_j(p,-h)$ by $\bar{v}_j(p)$ and $\bar{v}'_j(p)$ where $\bar{v}'_j(p,z)$ stands for the differentiation of $\bar{v}_j(p,z)$ with respect to z and eliminating $B(p)$ between them, we get

$$\bar{v}_2(p,z) = -\frac{\beta_2 \cosh\beta_2 z - \gamma\beta_1 \sinh\beta_2 z}{\beta_2(\beta_2 \sinh\beta_2 h + \gamma\beta_1 \cosh\beta_2 h)}\bar{v}'_2(p). \tag{19}$$

Putting $z = -h$ in (19), we find

$$\bar{v}_2(p) = -\frac{\beta_2 \cosh\beta_2 h + \gamma\beta_1 \sinh\beta_2 h}{\beta_2(\beta_2 \sinh\beta_2 h + \gamma\beta_1 \cosh\beta_2 h)}\bar{v}'_2(p). \tag{20}$$

We multiply (5) by $\exp(ipx)$ and integrate it between $x = 0$ and $x = \infty$ to find ($j = 2$)

$$\frac{d^2}{dz^2}[\bar{v}_{2+}(p,z)] - \beta_2^2 \bar{v}_{2+}(p,z) = \left(\frac{\partial v_2}{\partial x}\right)_{x=0} - ip(v_2)_{x=0}. \tag{21}$$

Changing p to $-p$ in (21) and subtracting the resulting equation from it, we get

$$\frac{d^2}{dz^2}[\bar{v}_{2+}(p,z) - \bar{v}_{2+}(-p,z)] - \beta_2^2[\bar{v}_{2+}(p,z) - \bar{v}_{2+}(-p,z)] = -2ip(v_2)_{x=0}. \tag{22}$$

Using the condition (12) in (22), we obtain

$$\frac{d^2}{dz^2}[\bar{v}_{2+}(p,z) - \bar{v}_{2+}(-p,z)] - \beta_2^2[\bar{v}_{2+}(p,z) - \bar{v}_{2+}(-p,z)] = 2ipA\cos\beta_{2N}(z+H). \tag{23}$$

The solution of (23) is

$$\bar{v}_{2+}(p,z) - \bar{v}_{2+}(-p,z) = E_1(p)\exp(-\beta_2 z) + E_2(p)\exp(\beta_2 z) - \frac{2ipA\cos\beta_{2N}(z+H)}{p^2 - k_{1N}^2}. \tag{24}$$

If the condition (11) is used, we find

$$\bar{v}_{2+}(p,z) - \bar{v}_{2+}(-p,z) = E(p)\cosh\beta_2(z+H) - \frac{2ipA\cos\beta_{2N}(z+H)}{p^2 - k_{1N}^2}. \tag{25}$$

Differentiating (25) with respect to z, it is obtained that

$$\bar{v}'_{2+}(p,z) - \bar{v}'_{2+}(-p,z) = E(p)\beta_2 \sinh\beta_2(z+H) + \frac{2ipA\beta_{2N}\sin\beta_{2N}(z+H)}{p^2 - k_{1N}^2}. \tag{26}$$

Putting $z = -h$ in (25) and (26) and eliminating $E(p)$ between them, we obtain

$$\bar{v}_{2+}(p) - \bar{v}_{2+}(-p) = \frac{\coth\beta_2\delta}{\beta_2}\left[\bar{v}'_{2+}(p) - \bar{v}'_{2+}(-p) - \frac{2ipA\beta_{2N}\sin\beta_{2N}(-h+H)}{p^2-k_{1N}^2}\right] - \frac{2ipA\cos\beta_{2N}(-h+H)}{p^2-k_{1N}^2}, \tag{27}$$

where $\delta = H - h$. Now eliminating $\bar{v}_{2+}(p)$ between (27) and (20), we find

$$\begin{aligned} &-\frac{F_1(p)\bar{v}'_{2+}(p)}{\beta_2 F_2(p)\sinh\beta_2\delta} + \frac{\coth\beta_2\delta iA\beta_{2N}\sin\beta_{2N}\delta}{\beta_2(p+k_{1N})} + \frac{iA\cos\beta_{2N}\delta}{p+k_{1N}} \\ &= \bar{v}_{2-}(p) - \bar{v}_{2+}(-p) + \frac{K_3(p)\bar{v}'_{2-}(p)}{\beta_2 F_2(p)} - \frac{\coth\beta_2\delta\bar{v}'_{2+}(-p)}{\beta_2} \\ &\quad - \frac{\coth\beta_2\delta iA\beta_{2N}\sin\beta_{2N}\delta}{\beta_2(p-k_{1N})} - \frac{iA\cos\beta_{2N}\delta}{p-k_{1N}}. \end{aligned} \tag{28}$$

This is the Wiener–Hopf type differential equation whose solution will give us $\bar{v}'_{2+}(p)$. $F_1(p)$, $F_2(p)$ and $K_3(p)$ are given by

$$\begin{aligned} F_1(p) &= \beta_2\sinh\beta_2 H + \gamma\beta_1\cosh\beta_2 H, \\ F_2(p) &= \beta_2\sinh\beta_2 h + \gamma\beta_1\cosh\beta_2 h, \\ K_3(p) &= \beta_2\cosh\beta_2 h + \gamma\beta_1\sinh\beta_2 h. \end{aligned} \tag{29}$$

Similarly, we obtain

$$\begin{aligned} &\frac{F_1(p)\bar{v}'_{2-}(p)}{\beta_2 F_2(p)\sinh\beta_2\delta} + \frac{\coth\beta_2\delta iA\beta_{2N}\sin\beta_{2N}\delta}{\beta_2(p-k_{1N})} + \frac{iA\cos\beta_{2N}\delta}{p-k_{1N}} \\ &= -\bar{v}_{2+}(p) - \bar{v}_{2-}(-p) + \frac{\coth\beta_2\delta}{\beta_2}\bar{v}'_{2-}(-p) - \frac{K_3(p)\bar{v}'_{2+}(p)}{\beta_2 F_2(p)} \\ &\quad - \frac{\coth\beta_2\delta iA\beta_{2N}\sin\beta_{2N}\delta}{\beta_2(p+k_{1N})} - \frac{iA\cos\beta_{2N}\delta}{p+k_{1N}}. \end{aligned} \tag{30}$$

This is the Wiener–Hopf type differential equation whose solution will give us $\bar{v}'_{2-}(p)$.

5. Factorization and decomposition

We factorize [9]

$$\left(\frac{\beta_2\delta}{\sinh\beta_2\delta}\right)\frac{F_1(p)}{F_2(p)} = P_+(p)P_-(p), \tag{31}$$

where

$$\begin{aligned} P_+(p) &= P_-(-p) = \frac{L_+(p)}{H_+(p)}\prod_{n=1}^{\infty}\frac{(p_+p_{1n})}{(p_+p_{2n})}, \\ H_+(p) &= H_-(-p), \quad L_+(p) = L_-(-p). \end{aligned} \tag{32}$$

$p = \pm p_{1n}$ and $p = \pm p_{2n}$ are the zeros of $F_1(p)$ and $F_2(p)$ respectively. $P_+(p)$ is analytic in the region $\beta > -k_1$ and $P_-(p)$ in $\beta < k_1$. Further

$$\sinh\beta_2\delta/\beta_2\delta = \prod_{n=1}^{\infty}[p^2\delta_n^2 + p_n^2\delta_n^2] = H_+(p)H_-(p), \tag{33}$$

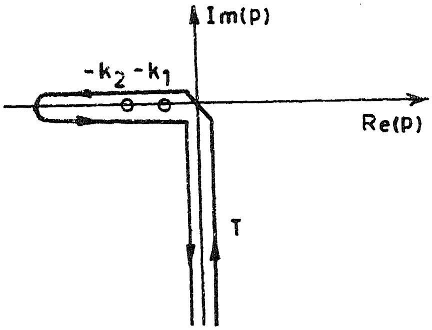

Figure 2. Contour of integration in the complex p-plane.

where

$$p_n \delta_n = (1 - k_2^2 \delta_n^2)^{1/2} = -i(k_2^2 \delta_n^2 - 1)^{1/2}, \quad \delta_n = \delta / n\pi. \tag{34}$$

$L_+(p)$ is given by [9] (figure 2)

$$\log L_+(p) = \frac{1}{\pi} \int_0^\infty \frac{\phi_1 - \phi_2}{u - ip} \mathrm{d}u + \frac{1}{\pi} \int_0^{k_1} \frac{\psi_1 - \psi_2}{u + p} \mathrm{d}u, \tag{35}$$

where

$$\tan\phi_1 = \gamma(u^2 + k_1^2)^{1/2} \cos((u^2 + k_2^2)^{1/2} H)/(u^2 + k_2^2)^{1/2} \sin((u^2 + k_2^2)^{1/2} H), \tag{36}$$

$$\tan\psi_1 = \gamma(k_1^2 - u^2)^{1/2} \cos((k_2^2 - u^2)^{1/2} H)/(k_2^2 - u^2)^{1/2} \sin((k_2^2 - u^2)^{1/2} H). \tag{37}$$

$\tan\phi_2$ and $\tan\psi_2$ are obtained replacing H by h in (36) and (37). $|P_\pm(p)|$ are asymptotic to $|p|^{1/2}$ as $|p| \to \infty$. We now decompose $\coth\beta_2\delta/\beta_2\delta$ as

$$\coth\beta_2\delta/\beta_2\delta = f_+(p) + f_-(p), \tag{38}$$

where

$$f_+(p) = f_-(-p) = -\frac{1}{2k_2\delta(p + k_2)} + \sum_{n=1}^{\infty} \frac{1}{p_n\delta(p + ip_n)}. \tag{39}$$

Also we write

$$K(p) = \frac{\beta_2 \cosh\beta_2 h + \gamma\beta_1 \sinh\beta_2 h}{\beta_2(\beta_2 \sinh\beta_2 h + \gamma\beta_1 \cosh\beta_2 h)} = \frac{K_3(p)}{\beta_2 F_2(p)}. \tag{40}$$

$K_3(p)/F_2(p)$ tends to 1 as $|p| \to \infty$. By infinite product theorem [8], $K_3(p)/F_2(p)$ can be factorized. Let $p = \pm p_{3n}$ be the zeros of $K_3(p)$, then

$$K_3(p)/F_2(p) = \prod_{n=1}^{\infty} \frac{(p^2 - p_{3n}^2)}{(p^2 - p_{2n}^2)} G_+(p) G_-(p), \tag{41}$$

where

$$\log G_+(p) = \frac{1}{\pi} \int_0^\infty \frac{N_1 - N_2}{u - ip} \mathrm{d}u - \frac{1}{\pi} \int_0^{k_1} \frac{M_1 - M_2}{u + p} \mathrm{d}u - \frac{1}{\pi} \int_{k_1}^{k_2} \frac{\mathrm{d}u}{u + p} \tag{42}$$

and $G_-(p) = G_+(-p)$ where

$$\tan N_1 = B_1 \cos B_1 h / \gamma (u^2 + k_1^2)^{1/2} \sin B_1 h,$$
$$\tan N_2 = \gamma (u^2 + k_1^2)^{1/2} \cos B_1 h / B_1 \sin B_1 h,$$
$$\tan M_1 = B_1' \cos B_1' h / \gamma (k_1^2 - u^2)^{1/2} \sin B_1' h,$$
$$\tan M_2 = \gamma (k_1^2 - u^2)^{1/2} \cos B_1' h / B_1' \sin B_1' h,$$
$$B_1 = (u^2 + k_2^2)^{1/2}, \quad B_1' = (k_2^2 - u^2)^{1/2}. \tag{43}$$

Thus

$$K(p) = \frac{G_+(p) G_-(p)}{(p + k_2)^{1/2} (p - k_2)^{1/2}} \prod_{n=1}^{\infty} \frac{(p \pm p_{3n})}{(p \pm p_{2n})} = K_+(p) K_-(p) \tag{44}$$

and $K_-(p) = -iK_+(-p)$. Equation (28) can now be written as

$$-\frac{\bar{v}_{2+}'(p) P_+(p)}{(p - k_2)(p + k_2)\delta} + \frac{iA\beta_{2N} \sin \beta_{2N}\delta [f_+(p) + f_-(p)]}{(p + k_{1N}) P_-(p)} + \frac{iA \cos \beta_{2N}\delta}{(p + k_{1N}) P_-(p)}$$
$$= \frac{\bar{v}_{2-}(p)}{P_-(p)} + \frac{\bar{v}_{2+}(-p)}{P_-(p)} + \frac{K_+(p) K_-(p) \bar{v}_{2-}'(p)}{P_-(p)} - [f_+(p) + f_-(p)]$$
$$\times \frac{\bar{v}_{2+}'(-p)}{P_-(p)} - [f_+(p) + f_-(p)] \frac{iA\beta_{2N} \sin \beta_{2N}\delta}{(p - k_{1N}) P_-(p)} - \frac{iA \cos \beta_{2N}\delta}{(p - k_{1N}) P_-(p)}. \tag{45}$$

From here, we write

$$-\frac{1}{(p - k_2)\delta} \left[\frac{\bar{v}_{2+}'(p) P_+(p)}{p + k_2} - \frac{\bar{v}_{2+}'(k_2) P_+(k_2)}{2k_2} \right]$$
$$-\frac{\bar{v}_{2-}'(-p_{2m}) K_+(p) K_-(-p_{2m})}{P_-(-p_{2m})} - \frac{\bar{v}_{2+}'(k_2)}{2k_2 \delta (p + k_2) P_-(-k_2)}$$
$$+ \sum_{n=1}^{\infty} \frac{i}{p_n \delta (p + ip_n) P_-(-ip_n)} \bar{v}_{2+}'(ip_n) + \frac{iA\beta_{2N} \sin \beta_{2N}\delta}{2k_2 \delta (p + k_2) P_-(-k_2)(k_2 + k_{1N})}$$
$$+ A\beta_{2N} \sin \beta_{2N}\delta \times \sum_{n=1}^{\infty} \frac{1}{p_n \delta (p + ip_n) P_-(-ip_n)(k_{1N} + ip_n)}$$
$$-\frac{iA\beta_{2N} \sin \beta_{2N}\delta}{2k_2 \delta (k_2 - k_{1N})(p + k_{1N}) P_-(-k_{1N})} + \frac{iA\beta_{2N} \sin \beta_{2N}\delta}{2k_2 \delta (k_2 - k_{1N})(p + k_2) P_-(-k_2)}$$
$$- A\beta_{2N} \sin \beta_{2N}\delta \sum_{n=1}^{\infty} \frac{1}{p_n \delta (k_{1N} - ip_n)(p + ip_n) P_-(-ip_n)} + \frac{A\beta_{2N} \sin \beta_{2N}\delta}{(p + k_{1N}) P_-(-k_{1N})}$$
$$\times \sum_{n=1}^{\infty} \frac{1}{p_n \delta (k_{1N} - ip_n)} + \frac{iA\beta_{2N} \sin \beta_{2N}\delta f_-(-k_{1N})}{(p + k_{1N}) P_-(-k_{1N})}$$
$$+ \frac{iA \cos \beta_{2N}\delta}{(p + k_{1N}) P_-(-k_{1N})} = O_-(p), \tag{46}$$

where $p_{2m} = k_2$, $p_{2n} \cdot O_-(p)$ includes the terms which are analytic in $\beta < d$ and the left hand member of (46) is analytic in $\beta > -d$. By analytic continuation, the two members represent an entire function as their regions of analyticity overlap. Each member tends

to zero in its region of analyticity as $|p|\to\infty$. By Liouville's theorem, the entire function is identically zero. Equating the left hand member of (46) to zero, we obtain

$$\bar{v}'_{2+}(p)=\left[\frac{T(p^2-k_2^2)}{p+k_{1N}}+\frac{\bar{v}'_{2+}(k_2)P_+(k_2)(p+k_2)}{2k_2}\right.$$
$$+\frac{i\bar{v}'_{2-}(-p_{2m})K_+(p_{2m})(p^2-k_2^2)\delta K_+(p)}{P_+(p_{2m})}-2iA\beta_{2N}\sin\beta_{2N}\delta(p^2-k_2^2)$$
$$\times\sum_{n=1}^{\infty}\frac{1}{(k_{1N}^2+p_n^2)(p+ip_n)P_+(ip_n)}-\frac{\bar{v}'_{2+}(k_2)(p-k_2)}{2k_2P_+(k_2)}$$
$$\left.+\sum_{n=1}^{\infty}\frac{i\bar{v}'_{2+}(ip_n)(p^2-k_2^2)}{p_n(p+ip_n)P_+(ip_n)}+\frac{iA\sin\beta_{2N}\delta(p-k_2)}{\beta_{2N}P_+(k_2)}\right]\frac{1}{P_+(p)},\tag{47}$$

where

$$T=\left[A\beta_{2N}\sin\beta_{2N}\delta\sum_{n=1}^{\infty}\frac{1}{p_n(k_{1N}-ip_n)}+iA\beta_{2N}\sin\beta_{2N}\delta f_+(k_{1N})\delta\right.$$
$$\left.-\frac{iA\beta_{2N}\sin\beta_{2N}\delta}{2k_2(k_2-k_{1N})}+iA\delta\cdot\cos\beta_{2N}\delta\right]\frac{1}{P_+(k_{1N})}.\tag{48}$$

Using (31), (38) and (44) in (30), we obtain

$$\frac{\bar{v}'_{2-}(p)P_-(p)}{\delta(p+k_2)(p-k_2)}+\frac{iA\beta_{2N}\sin\beta_{2N}\delta}{(p-k_{1N})P_+(p)}[f_+(p)+f_-(p)]+\frac{iA\cos\beta_{2N}\delta}{(p-k_{1N})P_+(p)}$$
$$=-\frac{\bar{v}_{2+}(p)}{P_+(p)}-\frac{\bar{v}_{2-}(-p)}{P_+(p)}+\frac{\bar{v}'_{2-}(-p)}{P_+(p)}[f_+(p)+f_-(p)]-\frac{K_+(p)K_-(p)\bar{v}'_{2+}(p)}{P_+(p)}$$
$$-\frac{iA\beta_{2N}\sin\beta_{2N}\delta}{(p+k_{1N})P_+(p)}[f_+(p)+f_-(p)]-\frac{iA\cos\beta_{2N}\delta}{(p+k_{1N})P_+(p)}.\tag{49}$$

From here, we write

$$\frac{1}{\delta(p+k_2)}\left[\frac{P_-(p)\bar{v}'_{2-}(p)}{p-k_2}+\frac{P_-(-k_2)\bar{v}'_{2-}(-k_2)}{2k_2}\right]-\frac{\bar{v}'_{2-}(-k_2)}{2k_2\delta(p-k_2)P_+(k_2)}$$
$$+\sum_{n=1}^{\infty}\frac{i\bar{v}'_{2-}(-ip_n)}{p_n(p-ip_n)P_+(ip_n)}+\frac{\bar{v}'_{2+}(p_{2m})K_-(p)K_+(p_{2m})}{P_+(p_{2m})}$$
$$+\frac{iA\beta_{2N}\sin\beta_{2N}\delta}{2k_2\delta(p-k_2)P_+(k_2)(k_2+k_{1N})}+A\beta_{2N}\sin\beta_{2N}\delta$$
$$\times\sum_{n=1}^{\infty}\frac{1}{p_n\delta(p-ip_n)(k_{1N}+ip_n)P_+(ip_n)}+\frac{iA\beta_{2N}\sin\beta_{2N}\delta f_+(k_{1N})}{(p-k_{1N})P_+(k_{1N})}$$
$$+\frac{iA\beta_{2N}\sin\beta_{2N}\delta}{2k_2\delta(k_2-k_{1N})(p-k_2)P_+(k_2)}-\frac{iA\beta_{2N}\sin\beta_{2N}\delta}{2k_2\delta(k_2-k_{1N})(p-k_{1N})P_+(k_{1N})}$$
$$+\frac{A\beta_{2N}\sin\beta_{2N}\delta}{(p-k_{1N})P_+(k_{1N})}\sum_{n=1}^{\infty}\frac{1}{p_n\delta(k_{1N}-ip_n)}+\frac{iA\cos\beta_{2N}\delta}{(p-k_{1N})P_+(k_{1N})}$$
$$-A\beta_{2N}\sin\beta_{2N}\delta\sum_{n=1}^{\infty}\frac{1}{p_n\delta(p-ip_n)(k_{1N}-ip_n)P_+(ip_n)}=R_+(p).\tag{50}$$

$R_+(p)$ includes the terms which are analytic in $\beta > -d$ and the left hand member of (50) is analytic in $\beta < d$. By analytic continuation, the two members represent an entire function as their regions of analyticity overlap. Each member tends to zero in its region of analyticity as $|p| \to \infty$. By Liouville's theorem, the entire function is identically zero. Equating the left hand member of (50) to zero, we get

$$\bar{v}'_{2-}(p) = \left[\frac{L(p^2-k_2^2)}{p-k_{1N}} - \frac{P_+(k_2)\bar{v}'_{2-}(-k_2)(p-k_2)}{2k_2} \right.$$
$$+ \frac{\bar{v}'_{2-}(-k_2)(p+k_2)}{2k_2P_+(k_2)} - \sum_{n=1}^{\infty} \frac{i\bar{v}'_{2-}(-ip_n)}{p_n(p-ip_n)P_+(ip_n)}(p^2-k_2^2)$$
$$- \frac{\bar{v}'_{2+}(p_{2m})(p^2-k_2^2)K_+(p_{2m})K_-(p)\delta}{P_+(p_{2m})} - \frac{iA\sin\beta_{2N}\delta(p+k_2)}{\beta_{2N}P_+(k_2)}$$
$$\left. + 2iA\beta_{2N}\sin\beta_{2N}\delta \sum_{n=1}^{\infty} \frac{(p^2-k_2^2)}{(k_{1N}^2+p_n^2)(p-ip_n)P_+(ip_n)}\right]\frac{1}{P_-(p)}, \tag{51}$$

where

$$L = \left[-iA\beta_{2N}\sin\beta_{2N}\delta f_+(k_{1N})\delta + \frac{iA\beta_{2N}\sin\beta_{2N}\delta}{2k_2(k_2-k_{1N})} - iA\delta\cdot\cos\beta_{2N}\delta \right.$$
$$\left. - A\beta_{2N}\sin\beta_{2N}\delta \sum_{n=1}^{\infty} \frac{1}{p_n(k_{1N}-ip_n)}\right]\frac{1}{P_+(k_{1N})}. \tag{52}$$

The displacement $v_2(x,z)$ is given by inversion of the Fourier transform i.e.

$$v_2(x,z) = \frac{1}{2\pi}\int_{-\infty+i\beta}^{\infty+i\beta} \bar{v}_2(p,z)\exp(-ipx)\mathrm{d}p \tag{53}$$

$$= \frac{1}{2\pi}\int_{-\infty+i\beta}^{\infty+i\beta} -\frac{1}{\beta_2}\left[\frac{\beta_2\cosh\beta_2 z - \gamma\beta_1\sinh\beta_2 z}{\beta_2\sinh\beta_2 h + \gamma\beta_1\cosh\beta_2 h}\right]\bar{v}'_2(p)\exp(-ipx)\mathrm{d}p \tag{54}$$

$$= \frac{1}{2\pi}\int_{-\infty+i\beta}^{\infty+i\beta} -\frac{1}{\beta_2}\left[\frac{\beta_2\cosh\beta_2 z - \gamma\beta_1\sinh\beta_2 z}{\beta_2\sinh\beta_2 h + \gamma\beta_1\cosh\beta_2 h}\right][\bar{v}'_{2+}(p) + \bar{v}'_{2-}(p)]$$
$$\times \exp(-ipx)\mathrm{d}p, \tag{55}$$

where $\bar{v}'_{2+}(p)$ and $\bar{v}'_{2-}(p)$ are given by (47) and (51) and $-d < \beta < d$.

6. Reflected and transmitted waves

We now evaluate the line integral (55) along a closed contour in the complex plane. The factor $\exp(-ipx) = \exp(-i\alpha x)\exp(\beta x)$ vanishes at infinity in the lower half plane if $x > 0$ and in the upper half plane if $x < 0$. Let us first evaluate the integral in the upper half plane (figure 3) where $x < 0$. There is a pole at $p = k_{1N}$ and the corresponding wave is given by

$$v_{2,1} = A_M\cos\beta_{2N}(z+H)\exp(-ik_{1N}x), \quad x < 0, \tag{56}$$

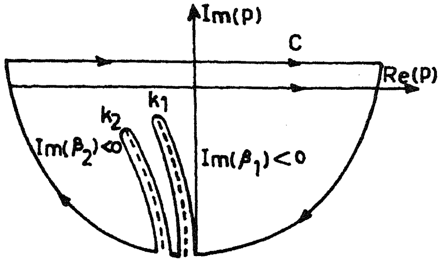

Figure 3. The contour of the integration in the complex p-plane with branch cuts.

where

$$A_M = \left[\frac{iP_+(k_2)\bar{v}'_{2-}(-k_2)(k_{1N}-k_2)}{2k_2} - \frac{i\bar{v}'_{2+}(p_{2m})K_-(k_{1N})K_+(p_{2m})}{P_+(p_{2m})}\cdot\beta_{2N}^2\delta\right.$$
$$-\frac{i\bar{v}'_{2-}(-k_2)(k_2+k_{1N})}{2k_2P_+(k_2)} - 2A\beta_{2N}^3\sin\beta_{2N}\delta$$
$$\times\sum_{n=1}^{\infty}\frac{1}{(k_{1N}^2+p_n^2)(k_{1N}-ip_n)P_+(ip_n)} + \sum_{n=1}^{\infty}\frac{\bar{v}'_{2-}(-ip_n)\beta_{2N}^2}{p_n(k_{1N}-ip_n)P_+(ip_n)}$$
$$\left.-\frac{A\sin\beta_{2N}\delta(k_{1N}+k_2)}{\beta_{2N}P_+(k_2)}\right]\frac{\sin\beta_{2N}\delta P_+(k_{1N})}{\beta_{2N}\delta\cos\beta_{2N}H(\mathrm{d}/\mathrm{d}p)[F_1(p)]_{p=k_{1N}}} \tag{57}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}p}[F_1(p)]_{p=k_{1N}} = k_{1N}\left[\frac{(\beta_{1N}H+\gamma)}{\beta_{1N}}\cos\beta_{2N}H + \frac{(1+\gamma\beta_{1N}H)}{\beta_{2N}}\sin\beta_{2N}H\right]. \tag{58}$$

The reflected Love waves are given by (56) in the region $-H\leqslant z\leqslant -h$, $x<0$ and their amplitudes are obtained from (57) by taking its modulus.

Let us now evaluate the integral (55) in the lower half plane where $x>0$. We see that $p=-ip_n$ are zeros of $\sinh\beta_2\delta$ and so $1/P_+(-ip_n)=0$. Also $1/P_-(ip_n)=0$ in the upper half plane where $x<0$. Therefore, $p=\pm ip_n$ are not poles of integrand. $p=-k_{1N}$ is a root of the equation

$$F_1(p) = \beta_2\sinh\beta_2 H + \gamma\beta_1\cosh\beta_2 H = 0, \tag{59}$$

i.e. $\tan\beta_{2N}H = \gamma\beta_{1N}/\beta_{2N}$.

The residue due to the pole at $p=-k_{1N}$ contributes

$$v_{2,2} = J_M\cos\beta_{2N}(z+H)\exp(ik_{1N}x),\ x>0,\ -H\leqslant z\leqslant 0, \tag{60}$$

where

$$J_M = \left[-\frac{i\bar{v}'_{2+}(k_2)P_+(k_2)(k_2-k_{1N})}{2k_2} - \frac{\bar{v}'_{2-}(-p_{2m})K_+(-k_{1N})K_+(p_{2m})}{P_+(p_{2m})}\right.$$
$$\times\beta_{2N}^2\delta - \frac{i\bar{v}'_{2+}(k_2)(k_2+k_{1N})}{2k_2P_+(k_2)} - \sum_{n=1}^{\infty}\frac{\bar{v}'_{2+}(ip_n)\beta_{2N}^2}{p_n(ip_n-k_{1N})P_+(ip_n)}$$

$$- 2A\beta_{2N}^3 \sin\beta_{2N}\delta \sum_{n=1}^{\infty} \frac{1}{(k_{1N}^2 + p_n^2)(k_{1N} - ip_n)P_+(ip_n)}$$

$$\left. - \frac{A\sin\beta_{2N}\delta(k_2 + k_{1N})}{\beta_{2N}P_+(k_2)}\right] \frac{P_+(k_{1N})\sin\beta_{2N}\delta}{\beta_{2N}\delta\cos\beta_{2N}H(\mathrm{d}/\mathrm{d}p)[F_1(p)]_{p=-k_{1N}}} \tag{61}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}p}[F_1(p)]_{p=-k_{1N}} = -\frac{\mathrm{d}}{\mathrm{d}p}[F_1(p)]_{p=k_{1N}}.$$

This represents the transmitted Love waves and their amplitudes are obtained from (61) by taking its modulus.

We again evaluate the integral (55) in the upper half plane where $x < 0$. Let p_{2n} be the root of the equation ($n = 1, 2, \dots$)

$$\beta_2 \sinh\beta_2 h + \gamma\beta_1 \cosh\beta_2 h = 0 \tag{62}$$

$$\text{i.e. } \tan\beta'_{2n}h = \gamma\beta'_{1n}/\beta'_{2n}, \tag{63}$$

where

$$\beta'_{1n} = (p_{2n}^2 - k_1^2)^{1/2}, \quad \beta'_{2n} = (k_2^2 - p_{2n}^2)^{1/2}. \tag{64}$$

The residue due to poles at $p = p_{2n}$ contributes

$$v_{2,3} = K_M \cos\beta'_{2n}(z+h)\exp(-ip_{2n}x), \quad x < 0, \quad -h \leqslant z \leqslant 0, \tag{65}$$

where

$$K_M = \left[\frac{iT\beta'^2_{2n}}{p_{2n} + k_{1N}} - \frac{i\bar{v}'_{2+}(k_2)P_+(k_2)(p_{2n} + k_2)}{2k_2} - \frac{\bar{v}'_{2-}(-p_{2m})K_+(p_{2n})}{P_+(p_{2m})}\right.$$

$$\times K_+(p_{2m})\beta'^2_{2n}\delta + \frac{i\bar{v}'_{2+}(k_2)(p_{2n} - k_2)}{2k_2 P_+(k_2)} - \sum_{n=1}^{\infty} \frac{\bar{v}'_{2+}(ip_n)\beta'^2_{2n}}{p_n(p_{2n} + ip_n)P_+(ip_n)}$$

$$+ 2A\beta_{2N}\beta_{2n}^2 \sin\beta_{2N}\delta \sum_{n=1}^{\infty} \frac{1}{(k_{1N}^2 + p_n^2)(p_{2n} + ip_n)P_+(ip_n)}$$

$$\left. + \frac{A\sin\beta_{2N}\delta(p_{2n} - k_2)}{\beta_{2N}P_+(k_2)}\right] \frac{1}{\cos\beta'_{2n}hP_+(p_{2n})(\mathrm{d}/\mathrm{d}p)[F_2(p)]_{p=p_{2n}}}, \tag{66}$$

$$\frac{\mathrm{d}}{\mathrm{d}p}[F_2(p)]_{p=p_{2n}} = \left[\frac{(h\beta'_{1n} + \gamma)}{\beta'_{1n}}\cos\beta'_{2n}h + \frac{(1 + \gamma\beta'_{1n})}{\beta'_{2n}}\sin\beta'_{2n}h\right]p_{2n}. \tag{67}$$

These are the reflected Love waves of nth mode in the surface layer with thickness h. If $h = 0$ i.e. there is no surface barrier, then $k_1 = p_{2n}$ such that the reflected waves propagate with a velocity equal to that of shear waves in the half space.

7. Scattered waves

There is a branch point $p = -k_1$ in the lower half plane. For contribution around this point, we put $p = -k_1 - iu$, u is small. The branch cut is obtained by taking $\mathrm{Re}(\beta_1) = 0$. Thus β_1 is imaginary and β_1^2 should be negative. Then

$$\beta_1^2 = p^2 - k_1^2 = (-k_1 - iu)^2 - k_1^2 = 2iu(k'_1 + ik''_1), \quad k_1 = k'_1 + ik''_1.$$

Taking $k'_1 = 0$

$$\beta_1 = \pm i(2k''_1 u)^{1/2} = \pm i\bar{\beta}_1, \tag{68}$$

$$\beta_2^2 = k_1^2 - k_2^2 + 2ik_1 u, \quad \text{or} \quad \beta_2 = (k_1^2 - k_2^2 + 2ik_1 u)^{1/2} = \bar{\beta}_2. \tag{69}$$

$\mathrm{Im}(\beta_1)$ has different signs on the two sides of the branch cut. Integrating (55) along two sides of the branch cut, we find

$$v_{2,4} = \frac{i}{2\pi}\int^{\infty} [[\bar{v}_2(p,z)]_{\beta_1 = i\bar{\beta}_1} - [\bar{v}_2(p,z)]_{\beta_1 = -i\bar{\beta}_1}]$$

$$\times \exp(-k_1'' x)\exp(-ux)\mathrm{d}u, \tag{70}$$

$$= -\frac{1}{\pi}\int_0^{\infty}\left[\frac{M(u)\gamma(2k_1''u)^{1/2}\cosh\bar{\beta}_2(z+H)}{(\bar{\beta}_2^2\sinh^2\bar{\beta}_2 H + \gamma^2\bar{\beta}_1^2\cosh^2\bar{\beta}_2 H)}\right.$$

$$\left. + \frac{Y(u)\gamma(2k_1''u)^{1/2}\cosh\bar{\beta}_2(z+h)}{(\bar{\beta}_2^2\sinh^2\bar{\beta}_2 h + \gamma^2\bar{\beta}_1^2\cosh^2\bar{\beta}_2 h)}\right]\exp(-k_1''x)\exp(-ux)\mathrm{d}u \tag{71}$$

$$= -\exp(-k_1''x)\int_0^{\infty} u^{1/2}F(u)\exp(-ux)\mathrm{d}u, \tag{72}$$

where

$$F(u) = -\frac{\gamma(2k_1'')^{1/2}}{\pi}\left[\frac{M(u)\cosh\bar{\beta}_2(z+H)}{\bar{\beta}_2^2\sinh^2\bar{\beta}_2 H + \gamma^2\bar{\beta}_1^2\cosh^2\bar{\beta}_2 H}\right.$$

$$\left. + \frac{Y(u)\cosh\bar{\beta}_2(z+h)}{\bar{\beta}_2^2\sinh^2\bar{\beta}_2 h + \gamma^2\bar{\beta}_1^2\cosh^2\bar{\beta}_2 h}\right], \tag{73}$$

$$M(u) = \left[\frac{T\bar{\beta}_2^2}{k_{1N} - k_1 - iu} + \frac{\bar{v}'_{2+}(k_2)P_+(k_2)(k_2 - k_1 - iu)}{2k_2}\right.$$

$$+ i\bar{v}'_{2-}(-p_{2m})\frac{K_+(-k_1 - iu)K_+(p_{2m})\bar{\beta}_2^2\delta}{P_+(p_{2m})}$$

$$+ \frac{\bar{v}'_{2+}(k_2)(k_2 + k_1 + iu)}{2k_2P_+(k_2)} + \sum_{n=1}^{\infty}\frac{i\bar{v}'_{2+}(ip_n)\bar{\beta}_2^2}{p_nP_+(ip_n)(ip_n - k_1 - iu)}$$

$$- 2iA\beta_{2N}\bar{\beta}_2^2\sin\beta_{2N}\delta\sum_{n=1}^{\infty}\frac{1}{(k_{1N}^2 + p_n^2)(ip_n - k_1 - iu)P_+(ip_n)}$$

$$\left. - \frac{iA\sin\beta_{2N}\delta}{\beta_{2N}P_+(k_2)}(k_2 + k_1 + iu)\right]P_-(-k_1 - iu)\frac{\sinh\bar{\beta}_2\delta}{\bar{\beta}_2\delta}, \tag{74}$$

$$Y(u) = \left[-\frac{L\bar{\beta}_2^2}{(k_1 + iu + k_{1N})} + \frac{\bar{v}'_{2-}(-k_2)(k_2 + k_1 + iu)}{2k_2}\right.$$

$$\times P_+(k_2) - \bar{v}'_{2+}(p_{2m})K_-(-k_1 - iu)\frac{K_+(p_{2m})\bar{\beta}_2^2\delta}{P_+(p_{2m})}$$

$$+ \frac{\bar{v}'_{2-}(-k_2)(k_2 - k_1 - iu)}{2k_2P_+(k_2)} + \sum_{n=1}^{\infty}\frac{i\bar{v}'_{2-}(-ip_n)\bar{\beta}_2^2}{p_n(k_1 + iu + ip_n)P_+(ip_n)}$$

$$- 2iA\beta_{2N}\bar{\beta}_2^2\sin\beta_{2N}\delta\sum_{n=1}^{\infty}\frac{1}{(k_{1N}^2 + p_n^2)(k_1 + iu + ip_n)P_+(ip_n)}$$

$$\left. - \frac{iA\sin\beta_{2N}\delta(k_2 - k_1 - iu)}{\beta_{2N}P_+(k_2)}\right]\frac{1}{P_-(-k_1 - iu)}. \tag{75}$$

The integral (72) can be evaluated using the result of Ewing *et al* [5]

$$\int_0^\infty u^{1/2}F(u)\exp(-ux)\mathrm{d}u=\left[\frac{F(0)\Gamma(3/2)}{x^{3/2}}+\frac{F'(0)\Gamma(5/2)}{x^{5/2}}+\frac{F''(0)\Gamma(7/2)}{x^{7/2}}+\cdots\right]. \tag{76}$$

$\Gamma(x)$ is a gamma function and retaining $F(0)$ only, we get from (72)

$$v_{2,4}=\frac{F(0)\Gamma(3/2)}{x^{3/2}}\exp(-k_1''x), \tag{77}$$

where

$$F(0)=-\frac{\gamma(2k_1'')^{1/2}}{\pi}\left[\frac{M(0)\cos\beta_2''(z+H)}{\beta_2''^2\sin^2\beta_2''H}+\frac{Y(0)\cos\beta_2''(z+h)}{\beta_2''^2\sin^2\beta_2''h}\right], \tag{78}$$

$$\beta_2''=(k_2^2-k_1^2)^{1/2},$$

$$v_{2,4}=\frac{\gamma(2k_1'')^{1/2}}{\Pi^{1/2}\beta_2''^2x^{3/2}}\left[\frac{M(0)(\exp(i\beta_2''(z-H))+\exp(-i\beta_2''(z+3H)))}{(1-\exp(-2i\beta_2''H))^2}\right.$$
$$\left.+\frac{Y(0)(\exp(i\beta_2''(z-h))+\exp(-i\beta_2''(z+3h)))}{(1-\exp(-2i\beta_2''h))^2}\right]\exp(-k_1''x). \tag{79}$$

The integral (53) vanishes when evaluated along two sides of the branch cut at $p=\pm k_2$.

8. Conclusion

The scattered waves in the relation (77) are of the form $[\cos\beta_2''(z+H)+\cos\beta_2''(z+h)]$ $\exp(-k_1''x)/x^{3/2}$ and decrease rapidly for large x. The waves in the half space can be obtained by evaluating $v_1(p,z)$. The reflected Love waves of nth mode in the surface

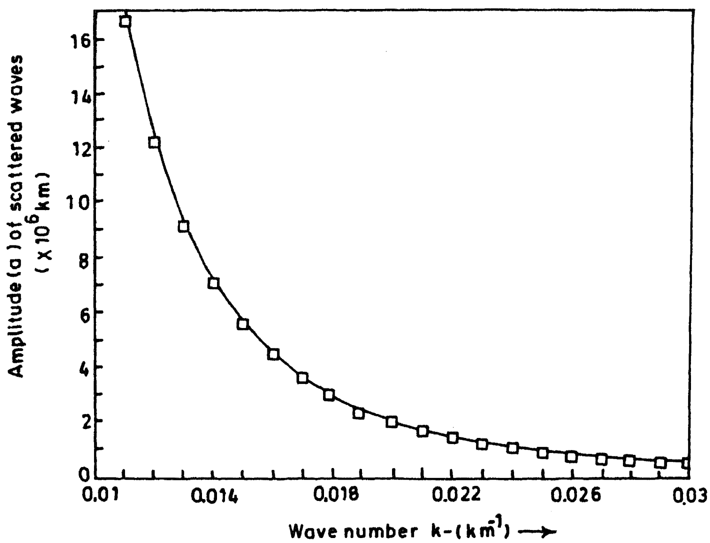

Figure 4. Graph between amplitude a of scattered wave and wave number k.

layer with thickness h are obtained from (65). The scattered waves propagate with the speed of the waves in the half space and not with that of the waves in the layer. Numerical computations are made by taking $z = -H$, $\gamma = \mu_1/\mu_2 = 2$, $H = 2\,\text{km}$, $h = 1{\cdot}6\,\text{km}$, $V_2/V_1 = 3/4$, $k_2 = k_{1N}$ and assuming $k_2\delta$ as very small. The amplitude of reflected wave decreases as the wave number increases. The amplitude of scattered wave has been plotted versus the wave number k as shown in figure 4. The amplitude of scattered wave falls off very rapidly as the wave number increases very slowly.

References

[1] Asghar S and Zaman F D, *Bull. Seis. Soc. Am.* **70** (1988) 241–257
[2] Deshwal P S, *Proc. Indian Natn. Sci. Acad.* **A50** (1984) 608–620
[3] Deshwal P S, *Geophys. Trans.* **33** (1988) 175–185
[4] Deshwal P S, *Acta Geophysica Polonica* **39** (1991)
[5] Ewing W M, Jardetsky W S and Press F, *Elastic waves in a layered media* (McGraw Hill Book Co.) (1957)
[6] Jones D S, *Q. J. Math.* (2) **3** (1952) 189–196
[7] Kazi M H, *Geophys. J. R. Astron. Soc.* **52** (1978) 25–44
[8] Noble B, *Methods Based on the Wiener–Hopf Technique* (Pergamon Press) (1958)
[9] Sato R, *J. Phys. Earth* **9** (1961) 19–36
[10] Xiao-Ping Li, *Wave motion* **22** (1995) 349–370
[11] Zaman F D, Asghar S and Khalid Hanif, *Punjab Univ. J. Math.* **24** (1991) 1–10

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 95–108.

On Ramanujan asymptotic expansions and inequalities for hypergeometric functions

R BALASUBRAMANIAN and S PONNUSAMY*

Institute of Mathematical Sciences, C.I.T. Campus, Chennai 600 113, India
*Department of Mathematics, Indian Institute of Technology, Institution of Engineers Building, Panbazar, Guwahati 781 001, India
Email: balu@imsc.ernet.in; samy@iitg.ernet.in

MS received 26 June 1997; revised 8 December 1997

Abstract. In this paper we first discuss refinement of the Ramunujan asymptotic expansion for the classical hypergeometric functions $F(a,b;c;x)$, $c \leqslant a+b$, near the singularity $x=1$. Further, we obtain monotonous properties of the quotient of two hypergeometric functions and inequalities for certain combinations of them. Finally, we also solve an open problem of finding conditions on $a,b>0$ such that

$$2F(-a,b;a+b;r^2) < (2-r^2)F(a,b;a+b;r^2)$$

holds for all $r \in (0,1)$.

Keywords. Hypergeometric functions; gamma function; elliptic integrals.

1. Introduction and main results

The Gaussian hypergeometric series (function) is defined by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!},$$

where a,b,c are complex numbers with $c \neq 0,-1,-2,\ldots$, $(a,0)=1$ for $a \neq 0$ and $(a,n+1) = (a+n)(a,n-1) = a(a+1)\cdots(a+n)$ for $n=0,1,2,\ldots$. In the exceptional case $c=-p$, $p=0,1,2,\ldots$, the function $F(a,b;c;z)$ is defined if $a=-m$ or $b=-m$, where $m=0,1,2,\ldots$ and $m \leqslant p$. The series $F(a,b;c;z)$ converges for $|z|<1$, and $F(a,b;c;z)$ can be continued analytically into the complex plane cut at $[1,\infty)$ (see [16]). The function $F(a,b;c;z)$ has a unique role among the special functions, since it is related to many other classes of special functions such as Bessel, Chebyshev, Legendre, Gegenbauer and Jacobi polynomials. Recall that $F(a,b;c;z)$ is called *zero-balanced* when $c=a+b$. In the special cases $a=1/2$ or $-1/2$, $b=1/2$ and $c=1$, we have

$$\mathscr{K}(x) = \frac{\pi}{2} F\left(\frac{1}{2},\frac{1}{2};1;x^2\right), \quad \mathscr{E}(x) = \frac{\pi}{2} F\left(-\frac{1}{2},\frac{1}{2};1;x^2\right), \quad 0<x<1,$$

and these functions are known as *Legendre's complete elliptic integrals of the first* and *second kind*, respectively. Set $\mathscr{K}'(x) = \mathscr{K}(x')$ and $\mathscr{E}'(x) = \mathscr{E}(x')$, $x' = \sqrt{1-x^2}$.

The basic identities due to Landen ([12], # 163·01, 164·02) (see also [11], p. 12 and [16], p. 507)

$$\mathscr{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathscr{K}(r), \quad \mathscr{K}\left(\frac{1-r}{1+r}\right) = \left(\frac{1+r}{2}\right)\mathscr{K}'(r)$$

give

$$\mu(r) = 2\mu\left(\frac{2\sqrt{r}}{1+r}\right), \quad \mu(r) = \frac{\pi}{2}\frac{F(1/2, 1/2; 1; 1-r^2)}{F(1/2, 1/2; 1; r^2)}. \tag{1.1}$$

Several inequalities dealing with $\mathscr{K}(r)$, $\mathscr{K}'(r)$, $\mathscr{E}(r)$, $\mathscr{E}'(r)$ and $\mu(r)$ have been derived in recent papers, and therefore it is natural to seek suitable restriction on the parameters a, b so that these inequalities are valid for the hypergeometric function $F(a, b; a+b; x^2)$. We observe that the generalized μ-function defined by

$$m(r) = \frac{F(a, b; a+b; 1-r^2)}{F(a, b; a+b; r^2)} \tag{1.2}$$

has brought new attention for obtaining additional applications of Ramanujan theory in the theory of modular equations [10]. Even though, in this paper, we are not going into the details of all such possible generalizations, we will point out some such results at the end of the paper.

Throughout the paper $B(a, b)$ denotes the Euler beta function

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{1.3}$$

provided the gamma function quotient is well defined. Further, we also use the notation

$$D(a, b, c) = \frac{B(c, a+b-c)}{B(a, b)}. \tag{1.4}$$

The motivation for the present study derives from the recent development on the Gauss–Ramanujan asymptotic formula reformulated in [1, 15]. If $c = a+b$, then as $x \to 1$ with $0 < x < 1$ we have the Ramanujan asymptotic formula ([9], p. 33–34) (see also [6], [13], Theorem 19)

$$F(a, b; a+b; x) = \frac{1}{B(a, b)}[R - \log(1-x) + O((1-x)\log(1-x))], \tag{1.5}$$

where

$$R := R(a, b) = -\psi(a) - \psi(b) - 2\gamma, \; \psi(a) = \Gamma'(a)/\Gamma(a), \tag{1.6}$$

and γ denotes the Euler–Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} k^{-1} - \log n\right) = 0{\cdot}57721566\ldots.$$

If $c < a+b$, then as $x \to 1$ with $0 < x < 1$ the asymptotic formula (see [16, p. 299]) is given by

$$F(a, b; c; x) \sim D(a, b, c)(1-x)^{c-a-b},$$

where $D = D(a, b, c)$ is defined by (1.4). Refined versions of the Ramanujan asymptotic relation (1.5) were obtained in [1, 15] ($c = a+b$) and a generalization to the case $c < a+b$ was given in [15]. We summarize these as follows.

1.7. Theorem. [1, 15] *For $a, b, c > 0$, let $B = B(a, b)$, $R = R(a, b)$ and $D = D(a, b, c)$ be as above. Then the following statements are true:*

(i) *The function* $f_1(x) \equiv (1 - F(a,b;c;x))/\log(1-x)$ *is strictly increasing from* $(0,1)$ *onto* $(ab/(a+b), 1/B)$.

(ii) *The function* $f_2(x) \equiv BF(a,b;a+b;x) + \log(1-x)$ *is strictly decreasing from* $(0,1)$ *onto* (R, B).

(iii) *The function* $f_3(x) \equiv BF(a,b;a+b;x) + (1/x)\log(1-x)$ *is increasing from* $(0,1)$ *onto* $(B-1, R)$ *if* $a \in (0, \infty)$ *and* $b \in (0, 1/a]$.

(iv) *The function* $f_3(x)$ *is decreasing from* $(0,1)$ *onto* $(R, B-1)$ *if* $a \in (1/3, \infty)$ *and* $b \geqslant (1+a)/(3a-1)$.

(v) *The function* $f_4(x) \equiv xF(a,b;a+b;x)/\log(1/(1-x))$ *is decreasing from* $(0,1)$ *onto* $(1/B, 1)$ *if* $a \in (0,\infty)$ *and* $b \in (0, 1/a]$.

(vi) *The function* $f_4(x)$ *is increasing from* $(0,1)$ *to the range* $(1, 1/B)$ *if* $a \in (1/2, \infty)$ *and* $b \geqslant a/(2a-1)$.

(vii) *For* $a > b$ *and* $c < a + b$, *we have*

(1) *The function* $g(x) \equiv F(a,b;c;x)/(1-x)^{c-a-b}, x \in (0,1)$, *is strictly increasing with range* $(1, D)$ *if* $c > a$ *or* $c < b$.

(2) *The function* $g(x)$ *is strictly decreasing from* $(0,1)$ *onto* $(D, 1)$ *if* $b < c < a$.

We observe that for $a \in (0, \infty)$ and $b \in (0, 1/a]$, Theorem 1.7 gives the following precise form of the Ramanujan approximation:

$$\begin{aligned} R(a,b) &< B(a,b)F(a,b;a+b;x) + \log(1-x) \\ &< R(a,b) + \frac{1-x}{x}\log\frac{1}{1-x} \quad \text{for} \quad x \in (0,1). \end{aligned}$$

For the proof of Theorem 1.7, an important fact used in [1, 15] was that all the coefficients of $F(a,b;a+b;x)$ should be positive. If, in the series $F(a,b;a+b;x)$, we choose a to be any complex quantity such that $\mathrm{Re}\,a > 0$ and b is the complex conjugate of a, then all the Maclaurin coefficients of the series $F(a,b;a+b;x)$ remain positive for $x \in (0,1)$. Since this case has not been handled in [1, 15], we first state our results, which are not covered in [1, 15], for this case.

1.8. Theorem. *Let a be a complex quantity such that* $\mathrm{Re}\,a > 0$ *and* $c > 0$.

(i) *The function* $g_1(x) \equiv F(a,\bar{a};c;x)/(1-x)^{c-2\mathrm{Re}\,a}$ *is increasing from* $(0,1)$ *with range* $(1, D(a,\bar{a},c))$ if $0 < c < 2\mathrm{Re}\,a$. *Increasing may be replaced by strictly increasing whenever* $c \neq a$.

(ii) *The function* $g_2(x) \equiv (F(a,\bar{a};c;x) - 1)/((1-x)^{c-2\mathrm{Re}\,a} - 1)$ *is increasing from* $(0,1)$ *onto*

$$\left(\frac{|a|^2}{c(2\mathrm{Re}\,a - c)}, D(a,\bar{a},c)\right)$$

if $0 < c < 2\mathrm{Re}\,a$. *Increasing may be replaced by strictly increasing if we also have* $c \neq a$.

1.9. Theorem. *Let* $B(a,b)$ *and* $R(a,b)$ *be defined by* (1.3) *and* (1.6), *respectively. Let* $a, b, c > 0$, *or* $b = \bar{a}$ *with a as a non-zero complex number and* $c > 0$. *Then we have*:

(i) *If* $c \geqslant \max\{0, a+b+ab\}$, *the function* $f(x) \equiv (1 - F(a,b;c;x))/\log(1-x)$ *is decreasing from* $(0,1)$ *onto* $(0, ab/c)$.

(ii) *If* $a \in \mathbb{C}$ *and* $c = 2\mathrm{Re}\,a > 0$, *the function* $f(x) \equiv (1 - F(a, \bar{a}; c; x))/\log(1 - x)$ *is strictly increasing from* $(0, 1)$ *onto* $(|a|^2/2\mathrm{Re}\,a, 1/B(a, \bar{a}))$, *and if* $0 < c < 2\mathrm{Re}\,a$, *it is strictly increasing from* $(0, 1)$ *onto* $(|a|^2/2\mathrm{Re}\,a, \infty)$.
(iii) *For* $a \in \mathbb{C}$ *such that* $\mathrm{Re}\,a > 0$, *the function* $g(x) \equiv B(a, \bar{a})F(a, \bar{a}; 2\mathrm{Re}\,a; x) + \log(1 - x)$ *is strictly increasing from* $(0, 1)$ *onto* $(R(a, \bar{a}), B(a, \bar{a}))$.

1.10 Theorem. *Define* $F(x) = F(a, b; c; x)$. *Let* $a, b, c > 0$, *or* $b = \bar{a}$ *with* a *as a non-zero complex number and* $c > 0$. *Then we have the following*:

(i) *If* $c(a + b + 2) \geqslant ab - 2$, *the function* $F''(x)/F'(x)$ *is increasing for* $x \in (0, 1)$.
(ii) *If* $c(a + b + 1) \geqslant ab$, *the function* $F'(x)/F(x)$ *is increasing for* $x \in (0, 1)$. *Strict inequality in each of the above two cases on* c *implies that the corresponding function is strictly monotone for* $x \in (0, 1)$.
(iii) *If* $\alpha, \beta > 0$ *and* $c \geqslant \max\{0, a + b - \beta, \alpha ab/\beta\}$, *the function* $(1 - x)^{\beta} F^{\alpha}(x)$ *is strictly decreasing for* $x \in (0, 1)$.
(iv) *If* $\alpha, \beta > 0$ *and* $c \leqslant \min\{0, a + b - \beta, \alpha ab/\beta\}$, *the function* $(1 - x)^{\beta} F^{\alpha}(x)$ *is strictly increasing for* $x \in (0, 1)$.

The special case $c = a + b$ of the following result has been used in [7] to solve a conjecture in [2], Problem 10, p. 80 and therefore, Theorem 1.11 will be a useful extension.

1.11. Theorem. *Define* $F(x) = F(a, b; c; x)$.

(1) *Suppose that* a *and* b *are related by any one of the following*:
  (i) $a, b > 0$ *and* $c \geqslant a + b$,
  (ii) $a, b \in (-1, 0)$ *and* $c > 0$,
  (iii) $a \in \mathbb{C} \setminus \{0\}$ *and* $0 \neq c \geqslant \max\{0, 2\mathrm{Re}\,a\}$.

Then, for $K = \max\{(ab + (a + b - c))/(c + 1), ab + 2(a + b - c)\}$, *the inequality*

$$x[(1 - x)F''(x) - F'(x)] < K[F(x) - 1]$$

holds for $x \in (0, 1)$.

(2) *Suppose that* a *and* b *are related by any one of the following*:
  (i) $a, b > 0$ *and* $0 < c < a + b$,
  (ii) $a \in \mathbb{C} \setminus \{0\}$ *and* $0 < c < 2\mathrm{Re}\,a$.

Then, for $K = \min\{(ab + (a + b - c))/(c + 1), ab + 2(a + b - c)\}$, *the inequality*

$$x[(1 - x)F''(x) - F'(x)] \geqslant K[F(x) - 1]$$

holds for $x \in (0, 1)$.

The proofs of Theorems 1.8–1.11 will be given in § 3.

The authors [2, 4, 5] proved several functional inequalities involving the hypergeometric function $F(a, b; c; x)$ with a, b, c real. The aim of such functional inequalities derived in these papers was to obtain certain generalized inequalities which were modeled after various inequalities for combinations of $\mathscr{K}(r)$ and $\mathscr{E}(r)$. In this connection, we again consider the situation where a is a complex quantity, $b = \bar{a}$ and c is a real quantity such that $c \neq 0, -1, -2, \ldots$. The case $c = 2\mathrm{Re}\,a$ with $\mathrm{Re}\,a > 0$ is more interesting because this choice covers also the behaviour of $\mathscr{K}(r)$. Thus, it is natural to look for

the extension in the neighbourhood of $(1/2, 1/2, 1)$ which deals with the case $\mathscr{K}(r)$. Now we state our next result, which gives new functional inequalities.

1.12. *Lemma. Let a be a complex quantity such that* $\mathrm{Re}\, a > 0$ *and* $F(x) = F(a, \bar{a}; 2\mathrm{Re}\, a; x)$. *Then we have the following*:

(i) *If* $3(\mathrm{Re} a)^2 + 6\mathrm{Re}\, a + 2 \geqslant (\mathrm{Im}\, a)^2$ *then the function* $F''(x)/F'(x)$ *is strictly increasing for* $x \in (0, 1)$.
(ii) *If* $3(\mathrm{Re} a)^2 + 2\mathrm{Re}\, a \geqslant (\mathrm{Im}\, a)^2$ *then the function* $F'(x)/F(x)$ *is strictly increasing for* $x \in (0, 1)$.
(iii) *For* $\alpha, \beta > 0$ *such that* $\alpha|a|^2 < 2\beta \mathrm{Re}\, a$, *the function* $(1-x)^{\beta} F^{\alpha}(x)$ *is strictly decreasing for* $x \in (0, 1)$.
(iv) *If* $3(\mathrm{Re} a)^2 + 2\mathrm{Re}\, a \geqslant (\mathrm{Im}\, a)^2$ *then the inequality* $F''(x)F(1-x) \leqslant F''(1-x)F(x)$ *holds for* $x \in (0, 1/2]$.
(v) *If* $\mathrm{Re}\, a > 0$ *then the inequality* $x[(1-x)F''(x) - F'(x)] < |a|^2[F(x) - 1]$ *holds for* $x \in (0, 1)$.

Proof. The cases (i)–(iii) follow from Theorem 1.10 whereas (iv) and (v) follow from Theorem 1.11. □□

1.13. Theorem. *Let a be a complex quantity such that* $0 < \mathrm{Re}\, a \leqslant 1$ *and that satisfies the condition*

$$3(\mathrm{Re} a)^2 + 2\mathrm{Re}\, a \geqslant (\mathrm{Im}\, a)^2.$$

If

$$m(r) = \frac{F(a, \bar{a}; 2\mathrm{Re}\, a; 1 - r^2)}{F(a, \bar{a}; 2\mathrm{Re}\, a; r^2)}, \tag{1.14}$$

then the function $m(\sqrt{1 - e^{-t}})$ *is decreasing and convex for* $t \in (0, \infty)$. *In other words, the function* $1/m(r)$ *is increasing and convex for* $r \in (0, 1)$. *In particular, we have the following inequality*:

$$\frac{1}{m(r)} + \frac{1}{m(s)} \geqslant \frac{2}{m(\sqrt{rs})}, \tag{1.15}$$

or, equivalently,

$$m(r) + m(s) \geqslant 2m\left(\sqrt{1 - \sqrt{(1-r^2)(1-s^2)}}\right) \tag{1.16}$$

hold for all $r, s \in (0, 1)$.

Proof. In the proof of Lemma 2.4 in [7], if we use Lemmas 2.1, 1.12 and Theorems 1.10 and 1.11, then after some computation we can easily see that the function

$$G(t) = \frac{F(e^{-t})}{F(1 - e^{-t})}, \quad \text{with} \quad F(x) = F(a, \bar{a}; 2\mathrm{Re}\, a; x)$$

is decreasing and convex for $t \geqslant 0$ whenever a is such that $0 < \mathrm{Re}\, a \leqslant 1$ and

$$3(\mathrm{Re} a)^2 + 2\mathrm{Re}\, a \geqslant (\mathrm{Im}\, a)^2.$$

The remaining part of the proof of the theorem follows easily by the same arguments which we have used for the proof of Theorem 1.9 in [7]. Thus we complete the proof. □□

2. Preliminary lemmas

Before establishing the main theorems, we need to prove some technical lemmas.

2.1. *Lemma.* [15] *Suppose that the power series* $f(z)=\Sigma_{n=0}^{\infty}a_n z^n$ *and* $g(z)=\Sigma_{n=0}^{\infty}b_n z^n$ *both converge for* $|z|<1$ *and that* $a_n\in\mathbb{R}$, $b_n>0$ *for all* $n\geqslant 0$. *Then* $f(x)/g(x)$ *is increasing(strictly) (decreasing (strictly)) for* $x\in(0,1)$ *if* a_n/b_n *is increasing (strictly) (decreasing (strictly)) for* $n\geqslant 0$.

A more general form of Lemma 2.1 has been presented in [15] and is one of the crucial facts in the proof of some of our main results.

2.2. *Lemma. For* $a\in\mathbb{C}\backslash\{0\}$ *and* $A>0$, *let* $Q(n)$ *be defined by*

$$Q(n)=\frac{|(a,n)|^2(n+A)}{(2\mathrm{Re}a,n)(1,n)},\quad n\geqslant 1.$$

(1) *If* A, a *are related by any one of the following conditions*:

(i) $A=|a|^2$ *and* $a\in\mathbb{C}\backslash\{1\}$,

(ii) $A<|a|^2$ *and* $a\in\mathbb{C}\backslash\{0\}$,

then the sequence $\{Q(n)\}$ *is strictly increasing to the limit* $1/B(a,\bar a)$. ($Q(n)=1$ *if* $a=1$.)

(2) *If* $A=2|a|^2/(2-|a-1|^2)$ *and* $|a-1|<\sqrt{2}$, *then the sequence* $\{Q(n)\}$ *is decreasing to the limit* $1/B(a,\bar a)$.

Proof. Define $\phi(n)=n(|a|^2-A)+(A+1)|a|^2-2A\mathrm{Re}a$. From the definition of $Q(n)$ it is easy to verify by simple computation that

$$Q(n+1)>Q(n)\Leftrightarrow\phi(n)>0,\quad\text{for}\quad n\geqslant 1.$$

(i) If $A=|a|^2$ and $a\in\mathbb{C}\backslash\{1\}$ then for all $n\geqslant 1$ we have $\phi(n)=|a|^2|a-1|^2>0$, and therefore the sequence $\{Q(n)\}$ is strictly increasing for $n\geqslant 1$.

(ii) First we assume that $A<|a|^2$ and $|a-1|\geqslant\sqrt{2}$. Then the coefficient of n in the expression $\phi(n)$ is positive and therefore, for all $n\geqslant 1$, we have

$$\phi(n)\geqslant\phi(1)=2|a|^2-A(2-|a-1|^2)\geqslant 2|a|^2>0.$$

(1) Suppose that $A<|a|^2$ and $|a-1|<\sqrt{2}$. Then in this case $\phi(1)>0$ provided $A<2|a|^2/(2-|a-1|^2)$, which is clearly true because of the assumption $A<|a|^2$. From these two observations, it follows that the sequence $\{Q(n)\}$ is strictly increasing for $n\geqslant 1$.

(2) Suppose that $A=2|a|^2/(2-|a-1|^2)$ and $|a-1|<\sqrt{2}$. Then we note that the condition on A implies that $A>|a|^2$, and therefore the coefficient of n in the expression $\phi(n)$ is negative so that

$$\phi(n)\leqslant\phi(1)=2|a|^2-A(2-|a-1|^2)=0.$$

Since

$$Q(n+1)\leqslant Q(n)\Leftrightarrow\phi(n)\leqslant 0,$$

the conclusion in this case follows from the fact that $\phi(n)\leqslant 0$ for all $n\geqslant 1$.

To find the limit of the sequence, we rewrite $Q(n)$ as

$$Q(n)=\frac{|a|^2}{2\mathrm{Re}a}\frac{|(a+1,n-1)|^2}{(2\mathrm{Re}a+1,n-1)(1,n-1)}+A\frac{|(a,n)|^2}{(2\mathrm{Re}a,n)(1,n)}.$$

Now we recall the following well-known result that follows easily from the Stirling formula:

$$\lim_{n\to\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)}=\begin{cases}\dfrac{\Gamma(c)}{\Gamma(a)\Gamma(b)} & \text{if} \quad c+1=a+b\\ 0 & \text{if} \quad c+1>a+b\\ \infty & \text{if} \quad c+1<a+b\end{cases}. \tag{2.3}$$

Finally, from (2.3), we deduce that $\lim_{n\to\infty}Q(n)=1/B(a,\bar{a})$. □□

When $a=1/2$ in parts 1 (i) and 2 of Lemma 2.2, we find that for each $n\geqslant 1$,

$$\frac{1}{\pi(n+2/7)}<\left(\frac{(1/2,n)}{(1,n)}\right)^2<\frac{1}{\pi(n+1/4)},$$

which improves both sides of the well-known Wallis inequalities that appear in ([14], p. 192, 3.1.16):

$$\frac{1}{\sqrt{\pi(n+1/2)}}<\frac{(1/2,n)}{(1,n)}<\frac{1}{\sqrt{\pi n}}.$$

Thus, Lemma 2.2 generalizes and improves the Wallis inequality in terms of complex parameters introduced through hypergeometric functions. Another generalization of Wallis inequality has recently been obtained in [1, 15].

3. Proofs of main results

3.1. *Proof of Theorem* 1.8. The idea of the proof is exactly as in [15] and so we just sketch the proof. Consider the sequence $\{Q(n)\}$, where

$$Q(n)=\frac{|(a,n)|^2}{(c,n)(2\mathrm{Re}\,a-c,n)}.$$

Using the ascending factorial notation $(a,n+1)=(a,n)(a+n)$, we may rewrite $Q(n)$ as

$$Q(n)=\left\{\frac{|a|^2}{2\mathrm{Re}\,a}\frac{(a+1,n-1)(\bar{a}+1,n-1)}{(2\mathrm{Re}\,a+1,n-1)(1,n-1)}\right\}$$
$$\times\frac{2\mathrm{Re}\,a(2\mathrm{Re}\,a+1,n-1)(1,n-1)}{c(2\mathrm{Re}\,a-c)(c+1,n-1)(2\mathrm{Re}\,a+1-c,n-1)}.$$

Using (2.3) we obtain the following:

$$\lim_{n\to\infty}Q(n)=\frac{1}{B(a,\bar{a})}\cdot B(c,2\mathrm{Re}\,a-c)\equiv D(a,\bar{a},c)\quad\text{for}\quad c<2\mathrm{Re}\,a.$$

Further, we easily get that

$$Q(n+1)\geqslant Q(n)\Leftrightarrow|c-a|^2\geqslant 0$$

and therefore $Q(n)$ is strictly increasing if and only if $|c-a|>0$. We note that $|c-a|=0$ if and only if a is a positive real number and $c=a$. Thus, the sequence $Q(n)$ is increasing to the limit $D(a,\bar{a},c)$, as $n\to\infty$. In particular, for each positive integer n,

$$\frac{|a|^2}{c(2\mathrm{Re}\,a-c)}<Q(n)<D(a,\bar{a},c). \tag{3.2}$$

Next we consider

$$F(a,b;c;z)=\sum_{n\geqslant 0}\alpha_n z^n \quad\text{and}\quad (1-z)^{c-2\mathrm{Re}a}=\sum_{n\geqslant 0}\beta_n z^n$$

and from the above series expansions we find that $Q(n)=\alpha_n/\beta_n$. Further, in the series expansion,

$$F(a,\bar{a};c;x)-D(a,\bar{a},c)(1-x)^{c-2\mathrm{Re}a}$$
$$=\sum_{n=0}^{\infty}\left(\frac{|(a,n)|^2}{(c,n)(1,n)}-D(a,\bar{a},c)\frac{(2\mathrm{Re}a-c,n)}{(1,n)}\right)x^n$$

all coefficients are non-negative (negative) according as $Q(n)$ is increasing (strictly increasing). Therefore, the conclusion is an immediate consequence of the monotonous properties of $Q(n)$ and Lemma 2.1. Thus we obtain that if $0<c<2\mathrm{Re}a$, then the function $g_1(x)\equiv F(a,\bar{a};c;x)/(1-x)^{c-2\mathrm{Re}a}$ is increasing from $(0,1)$ onto the range $(1,D(a,\bar{a},c))$ and is strictly increasing if we also have $c\neq a$.

Part (ii) follows similarly. □□

3.3. *Proof of Theorem* 1.9. For $n\geqslant 1$, we define

$$\alpha_n=\frac{(a,n)(b,n)}{(c,n)(1,n)},\quad \beta_n=\frac{1}{n}\quad\text{and}\quad Q(n)=\frac{\alpha_n}{\beta_n}.$$

Then for $|z|<1$, we can write $F(a,b;c;z)-1=\Sigma_{n\geqslant 1}\alpha_n z^n$ and $-\log(1-z)=\Sigma_{n\geqslant 1}\beta_n z^n$. Using (2.3), we easily obtain

$$\lim_{n\to\infty}Q(n)=\begin{cases}\dfrac{1}{B(a,b)} & \text{if } c=a+b\\ 0 & \text{if } c>a+b\\ \infty & \text{if } c<a+b\,.\end{cases}$$

Simple calculation yields that

$$Q(n+1)>Q(n)\Leftrightarrow n(a+b-c)+ab>0$$

and

$$Q(n+1)\leqslant Q(n)\Leftrightarrow n(a+b-c)+ab\leqslant 0.$$

From the above observations, it can be easily seen that the conclusion for each case follows from the method of proof of Theorem 1.8, from the respective conditions on a,b,c and from Lemma 2.1. Therefore we omit the details. □□

3.4. *Proof of Theorem* 1.10. Let $F(x)=F(a,b;c;x)$. From the definition of the hypergeometric series, we easily obtain the derivative formula for $F(x)$:

$$F'(x)=\frac{ab}{c}F(a+1,b+1;c+1;x) \tag{3.5}$$

and

$$F''(x)=\frac{(a,2)(b,2)}{(c,2)}F(a+2,b+2;c+2;x).$$

For convenience, we let $F(x)=\Sigma_{n=0}^{\infty}A_n x^n$, $F'(x)=\Sigma_{n=0}^{\infty}B_n x^n$ and $F''(x)=\Sigma_{n=0}^{\infty}C_n x^n$. Therefore, we can write

$$A_n=\frac{(a,n)(b,n)}{(c,n)(1,n)},\quad B_n=\frac{(a,n+1)(b,n+1)}{(c,n+1)(1,n)},\quad C_n=\frac{(a,n+2)(b,n+2)}{(c,n+2)(1,n)}$$

so that by a simple calculation we have

$$\frac{C_n}{B_n}=\frac{(a+n+1)(b+n+1)}{(c+n+1)}\quad\text{and}\quad\frac{B_n}{A_n}=n+a+b-c+\frac{(c-a)(c-b)}{n+c}. \tag{3.6}$$

(i) Now we assume that either a,b,c are all positive real numbers, or a is a non-zero complex number such that $b=\bar{a}$ and $c>0$. Therefore by a simple computation we can easily find that

$$\frac{C_n}{B_n}\leqslant\frac{C_{n+1}}{B_{n+1}}\Leftrightarrow\phi(n)\leqslant 0, \tag{3.7}$$

where

$$\phi(n)=n^2+(2c+3)n+c(a+b+3)+2-ab. \tag{3.8}$$

We remark that the inequality (3.7) continues to hold if we replace both inequalities in (3.7) by strict inequalities, respectively. Since $c>0$, the function $\phi(n)$ is increasing for $n\geqslant 0$, and therefore, it follows that

$$\phi(n)\geqslant\phi(0)=c(a+b+3)+2-ab.$$

This observation shows that, if a,b,c are related by the condition $c(a+b+3)+2-ab\geqslant 0$ then the sequence $\{C_n/B_n\}$ is increasing for all $n\geqslant 0$, and hence, by Lemma 2.1, it follows that the function $F''(x)/F'(x)$ is increasing for $x\in(0,1)$.

(ii) Again, by Lemma 2.1, it suffices to show that the ratio of the coefficients B_n/A_n is strictly decreasing for $n\geqslant 0$. By a simple computation, we note that the inequality

$$\frac{B_n}{A_n}<\frac{B_{n+1}}{A_{n+1}}$$

is equivalent to

$$(n+c)(n+c+1)>(c-a)(c-b).$$

Since $c>0$, the last inequality holds for all $n\geqslant 0$ if it holds for $n=0$. Putting $n=0$ in the last inequality we have $c(c+1)>(c-a)(c-b)$, which is equivalent to $c(1+a+b)>ab$. Therefore the conclusion follows from Lemma 2.1 if a,b,c are related by $c(1+a+b)>ab$ and we complete the proof.

(iii) Consider the function $f(x)=(1-x)^{\beta}(F(a,b;c;x))^{\alpha}$. Then using the derivative formula for $F(a,b;c;x)$, we have

$$f'(x)=(1-x)^{\beta-1}(F(a,b;c;x))^{\alpha-1}$$
$$\times\left[-\beta F(a,b;c;x)+\frac{\alpha ab}{c}(1-x)F(a+1,b+1;c+1;x)\right].$$

Using the series expansion for the square bracketed term of the above expression, we can write

$$f'(x) = (1-x)^{\beta-1}(F(a,b;c;x))^{\alpha-1}$$
$$\times \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n+1)(1,n)} [\alpha ab - \beta c + (a+b-c-\beta)n] x^n.$$

The conditions on c, i.e., $\alpha, \beta, a, b, c > 0$ and $\alpha ab < \beta c$, imply that

$$\alpha ab - \beta c + (a+b-c+\beta)n \leqslant \alpha ab - \beta c \leqslant 0$$

for all $n \geqslant 0$. This observation shows that $f'(x) < 0$ for $x \in (0,1)$ and therefore, the function f is decreasing for $x \in (0,1)$.
(iv) Follows from part (iii) and from the fact that the given condition on c implies that

$$\alpha ab - \beta c + (a+b-c-\beta)n \geqslant 0$$

for all $n \geqslant 0$. □□

3.9. *Proof of Theorem* 1.11. (i) Define $F(x) = F(a,b;c;x)$, where $a, b, c > 0$ and $c \geqslant a+b$. Then from the series expansions for $F'(x)$ and $F''(x)$, we easily compute that

$$(1-x)F''(x) - F'(x)$$
$$= \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{(c,n+2)(1,n)} [n(a+b-c) + a + b + ab - c] x^n,$$

so that using the series expansion for $F(x)$ and simplification, we find that the inequality

$$x[(1-x)F''(x) - F'(x)] < K[F(x) - 1] \tag{3.10}$$

is equivalent to the inequality

$$\sum_{n=1}^{\infty} \frac{(a,n)(b,n)}{(c,n+1)(1,n)} \{\phi(n)\} x^n > 0,$$

where

$$\phi(n) = n^2(c-a-b) + n(K-ab) + Kc,$$

and K is defined in Theorem 1.11. We divide the proof into two parts.

Let $c = a+b$. Then in this case $K = ab$ so that for all $n \geqslant 1$ we have $\phi(n) = cab = (a+b)ab > 0$ and therefore (3.10) holds since $a, b > 0$.

Next, we assume $c > a+b$. Clearly for large n, $\phi(n) > 0$. Now, for all $n \geqslant 1$, the condition on c and K gives

$$\phi'(n) = 2n(c-a-b) + (K-ab) \geqslant 2(c-a-b) + (K-ab) \geqslant 0,$$

so that $\phi(n)$ is an increasing function of n. Therefore, using the condition on K, we deduce that

$$\phi(n) \geqslant \phi(1) = K(1+c) + c - a - b - ab \geqslant 0$$

for $n \geqslant 1$. This observation shows that the inequality (3.10) holds under the given condition on K.

The other parts may be checked similarly. □□

4. Concluding remarks

In this section we first state a few preliminary results in the form of a proposition which extends Theorem 1.7 in ([3], eq. (18)), Theorems 1.1(2) and 2.1(6) in [4].

4.1. PROPOSITION

(i) *For $a, b > 0$ and $x \in (0,1)$, we have*

$$F(a,b;2b;x) < (1+x)^{2a} F\left(a, a-b+\frac{1}{2}; b+\frac{1}{2}; x^2\right). \tag{4.2}$$

(ii) *For $a \in \mathbb{R}$, $b, c \in (0,\infty)$ and $x \in (0,1)$,*

$$F(a,b;c;x) + F(-a,b;c;x) \geqslant 2.$$

(iii) *For $a, b, c \in (0,\infty)$ and $x \in (0,1)$ the function*

$$\frac{F(a,b;c;x) - F(-a,b;c;x)}{x}$$

is strictly increasing and convex on $(0,1)$ and has the limit $2ab/c$ as $x \to 0$.

Proof. (i) Recall the Gauss transformation

$$F\left(a,b;2b;\frac{4x}{(1+x)^2}\right) = (1+x)^{2a} F\left(a, a-b+\frac{1}{2}; b+\frac{1}{2}; x^2\right), \quad x \in (0,1), \tag{4.3}$$

where $2b \neq 0, -1, -2, \ldots$ (see [8], p. 111, eq. (5)) and also ([9], Entry 3 in ch. 11, p. 50). Proof of (4.2) follows from (4.3) since $x < 4x/(1+x)^2$ and since the function $F(a,b;2b;x^2)$ is increasing on $[0,1)$.
(ii) Suppose that $a \in \mathbb{R}$ and $b, c \in (0,\infty)$. Now, we can write

$$F(a,b;c;x) + F(-a,b;c;x) - 2 = \sum_{n=2}^{\infty} \frac{(b,n)}{(c,n)(1,n)} [(a,n) + (-a,n)] x^n, \quad x \in (0,1). \tag{4.4}$$

Using the triangle inequality we see that

$$|(-a,n)| = |a||-a+1| \cdots |-a+n-1| \leqslant |a|(|a|+1) \cdots (|a|+n-1) = (|a|,n).$$

This observation shows that all the coefficients of the power series of the function (4.4) are positive and therefore the conclusion follows.
(iii) Suppose that $a, b, c \in (0,\infty)$. Now, we can write

$$\frac{F(a,b;c;x) - F(-a,b;c;x)}{x} = \frac{2ab}{c} + \sum_{n=2}^{\infty} \frac{(b,n)}{(c,n)(1,n)} [(a,n) - (-a,n)] x^{n-1}. \tag{4.5}$$

As in the proof of part (ii), the triangle inequality immediately gives $|(-a,n)| \leqslant (|a|,n) = (a,n)$ so that $(a,n) - (-a,n) > 0$ for $a > 0$ and all $n \geqslant 2$. Thus, all the coefficients of the power series of the function (4.5) are positive and the constant term is $2ab/c$ and the conclusion follows. □□

In ([5], Problem 9, p. 80), the authors state another problem which is based on the inequality

$$2\mathscr{E}(r) < (2-r^2)\mathscr{K}(r), \quad 0<r<1.$$

4.6. *Problem.* Is it true that for $a, b > 0$

$$2F(-a, b; a+b; r^2) < (2-r^2)F(a, b; a+b; r^2)?$$

Clearly, the above inequality is not true for a close to zero. The answer to Problem 4.6 will be divided into three different parts which are as follows.

4.7. Theorem. *Let $a \in (0, 1]$ and $b > 0$. Then a necessary and sufficient condition for*

$$2F(-a, b; a+b; r^2) < (2-r^2)F(a, b; a+b; r^2), \quad r \in (0, 1), \tag{4.8}$$

is that $a \in (1/4, 1]$ and $b \in [a/(4a-1), \infty)$.

Proof. For convenience, we let $r^2 = t > 0$. Writing (4.8) as

$$(1-t/2)F(a, b; a+b; t) - F(-a, b; a+b; t) > 0$$

and then using the series expansion for $F(a, b; c; t)$, we easily see that inequality (4.8) is equivalent to

$$B_1 t + \sum_{n=2}^{\infty} B_n t^n > 0, \tag{4.9}$$

where $B_1 = (2ab/(a+b)) - 1/2$ and, for $n \geqslant 2$,

$$B_n = \frac{(a, n-1)(b, n-1)}{2(a+b, n)(1, n)}[(n-1)(n-2)$$
$$+(n-2)(a+b) + 2ab] - \frac{(-a, n)(b, n)}{(a+b, n)(1, n)}.$$

Suppose that $a \in (0, 1)$. Then the coefficient of t^n for $n \geqslant 2$ is clearly positive and therefore (4.9) holds if $B_1 \geqslant 0$, which is equivalent to $b(4a-1) \geqslant a$. Since $0 < a \leqslant 1$, the last inequality requires that a has to be greater than $1/4$ so that the condition on b becomes $b \geqslant a/(4a-1)$. This proves the sufficient part.

Next we prove the necessity part. In this case, dividing the inequality (4.9) by t and then taking the limit as $t \to 0$, it follows that the first coefficient B_1 has to be non-negative, i.e., $b(4a-1) \geqslant a$ is a necessary condition for the truth of the inequality (4.8). □□

Our next theorem considers the case $a > 1$.

4.10. Theorem. *Let $a \in (1, \infty)$ and $b > 0$. If a, b are related by any one of the following*

(i) *$a \in (1, (3+\sqrt{5})/4)$ and $b \in [a/(4a-1), \infty)$,*
(ii) *$a \in [(3+\sqrt{5})/4, \infty)$ and $b \in [a-1, \infty)$,*

then the inequality (4.8) holds.

Proof. The idea and the notation are as in the proof of Theorem 4.7. Let $a > 1$ and B_n be defined as in Theorem 4.7. Then after some computation we find that $B_n > 0$ for $n \geqslant 2$ is equivalent to the inequality

$$H(a) > 0, \tag{4.11}$$

where

$$H(a) = 2(a-1)(b+n-1)[(a+1, n-2) - (-a+2, n-2)] + n(a+1, n-1)(n+b-a-1).$$

(i) Suppose that the assumption (i) holds. Then the inequality

$$\frac{a}{4a-1} > a-1$$

holds and in particular $B_1 > 0$ and $b > a-1$. It is trivial to see that for $a > 1/2$ and $n \geqslant 2$, the inequality

$$(a+1, n-2) - (-a+2, n-2) > 0$$

holds. The condition $b > a-1$ and the fact that $a > 1$ imply that (4.11) holds so that $B_n > 0$ for all $n \geqslant 2$. This observation shows that if a and b are related by (i) then the inequality (4.8) holds.

(ii) Assume that (ii) holds. Then in this case we have

$$\frac{a}{4a-1} \leqslant a-1$$

and since $b \geqslant a-1$, we see that $B_n > 0$ for $n \geqslant 1$ and the conclusion follows similarly.

Acknowledgements

The authors are thankful to the referee for his useful comments. This work was completed during the second author (SP)'s visit to The Institute of Mathematical Sciences, Chennai and he would also like to thank Prof. Matti Vuorinen for helpful discussion during his visit to the University of Helsinki. This work was partially supported by National Board for Higher Mathematics, India.

References

[1] Anderson G D, Barnard R W, Richards K C, Vamanamurthy M K and Vuorinen M, Inequalities for zero-balanced hypergeometric functions, *Trans. Am. Math. Soc.* **347** (1995) 1713–1723

[2] Anderson G D, Vamanamurthy M K and Vuorinen M, Special functions of quasiconformal theory, *Exposition. Math.* 7 (1989) 97–136

[3] Anderson G D, Vamanamurthy M K and Vuorinen M, Functional inequalities for complete elliptic integrals and their ratios, *SIAM J. Math. Anal.* **21(2)** (1990) 536–549

[4] Anderson G D, Vamanamurthy M K and Vuorinen M, Functional inequalities for hypergeometric functions and complete elliptic integrals, *SIAM J. Math. Anal.* **23(2)** (1992) 512–524

[5] Anderson G D, Vamanamurthy M K and Vuorinen M, *Hypergeometric functions and elliptic integrals in current topics in analytic function theory,* edited by H M Srivastava and S Owa (Singapore–London: World Sci. Publ. Co.) (1992) pp. 48–85

[6] Askey R, *S Ramanujan and hypergeometric and basic hypergeometric series* (*Russian*), translated from English and with a remark by N M Atakishiev and S K Suslov, *Uspekhi Mat. Nauk.* **45(1)(271)** (1990) 33–76, 222; translation in *Russian Math. Surveys* **451** (1990) 37–86

[7] Balasubramanian R, Ponnusamy S and Vuorinen M, Functional inequalities for the quotients of hypergeometric functions, *J. Math. Anal. Appl.* **218** (1998) 256–268
[8] Bateman H, *Higher transcendental functions* edited by A Erdelyi, W Magnus, F Oberhettinger and F G Tricomi (New York: McGraw-Hill) (1953) vol. I
[9] Berndt B C, *Ramanujan's Notebooks* (New York: Springer-Verlag) (1987) part II
[10] Berndt B C, Bhargava S and Garvan F G, Ramanujan's theories of elliptic functions to alternative bases, *Trans. Am. Math. Soc.* **347** (1995) 4163–4244
[11] Borwein J M and Borwein P B, Inequalities for compound mean iterations with logarithmic asymptotes, *J. Math. Anal. Appl.* **177** (1993) 572–582
[12] Byrd P F and Friedman M D, Handbook of elliptic integrals for engineers and physicists, *Die Grundlehren der math. Wissenschaften* (Berlin-Göttingen-Heidelberg: Springer-Verlag) (1954) vol. 57
[13] Evans R J, Ramanujan's second notebook: Asymptotic expansions for hypergeometric series and related functions, in *Ramanujan Revisited: Proc. of the Centenary Conference, University of Illinois at Urbana-Champaign* edited by G E Andrews, R A Askey, B C Berndt, R G Ramanathan and R A Rankin (Boston: Academic Press) (1988) pp. 537–560
[14] Mitrinovic D S, Analytic inequalities, *Die Grundlehren der math. Wissenschaften,* Band 165 (Springer-Verlag) (1970)
[15] Ponnusamy S and Vuorinen M, Asymptotic expansions and inequalities for hypergeometric functions, *Mathematika* **44** (1997) 278–301
[16] Whittaker E T and Watson G N, *A Course of Modern Analysis*, 4th ed. (Cambridge University Press) (1958)

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 109–120.

Degree of approximation of functions associated with Hardy–Littlewood series in the generalized Hölder metric

G DAS, A K OJHA* and B K RAY*†

Department of Mathematics, Utkal University, Vanivihar, Bhubaneswar 751 004, India
*Department of Mathematics, Ravenshaw College, Cuttack, 753 003, India
†Mailing Address: Plot No. 102, Saheed Nagar, Bhubaneswar 751 007, India

MS received 4 December 1996; revised 16 December 1997

Abstract. The paper studies the degree of approximation of functions associated with Hardy Littlewood series in the generalized Hölder metric.

Keywords. Banach space; Hölder metric; generalized Hölder metric, infinite matrix; Hardy Littlewood series.

1. Definition

Let f be a periodic function of period 2π and let $f \in L_p\,[0, 2\pi]$ for $p \geqslant 1$. Then the Fourier series of f at $t = x$ is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x), \tag{1.1}$$

and conjugate series by

$$\sum (a_n \sin nx - b_n \cos nx) = \sum_{n=1}^{\infty} B_n(x). \tag{1.2}$$

Let us write

$$\Phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}, \tag{1.3}$$

$$\chi_x(u) = \int_u^{\pi} \Phi_x(w)\frac{1}{2}\cot w/2 \,\mathrm{d}w. \tag{1.4}$$

Let $s_n(x)$ and $s_n^*(x)$ respectively denote the partial sum and modified partial sum of (1.1), i.e.

$$s_n(x) = \sum_{k=0}^{n} A_k(x), \quad s_n^*(x) = \sum_{k=0}^{n-1} A_k(x) + \frac{1}{2}A_n(x).$$

It is known ([9], p. 50) that

$$s_n^*(x) - f(x) = \frac{2}{\pi}\int_0^{\pi} \frac{\Phi_x(u)\sin nu}{2\tan u/2}\,\mathrm{d}u. \tag{1.5}$$

The space $L_p\,[0, 2\pi]$ where $p = \infty$ includes the space $C_{2\pi}$ of all continuous functions defined over $[0, 2\pi]$.

We write

$$\|f\|_c = \sup_{x\in[0,2\pi]} |f(x)|$$

$$= \|f\|_p \quad (p=\infty)$$

and for $p \geqslant 1$,

$$\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p \,dx\right)^{1/p} \quad (p \geqslant 1).$$

We write

$$\omega(\delta) = \omega(\delta, f) = \sup_{0\leqslant h\leqslant\delta} \|f(\cdot+h) - f(\cdot)\|_c, \tag{1.6}$$

when the norm has been taken with respect to x throughout the paper

$$\omega_p(\delta) = \omega_p(\delta, f) = \sup_{0\leqslant h\leqslant\delta} \|f(\cdot+h) - f(\cdot)\|_p, \tag{1.7}$$

$$\omega_p^{(2)}(\delta) = \omega_p^{(2)}(\delta, f) = \sup_{0\leqslant h\leqslant\delta} \|f(\cdot+h) + f(\cdot-h) - 2f(\cdot)\|_p, \tag{1.8}$$

which are respectively called modulus of continuity, integral modulus of continuity and integral modulus of smoothness (see [9] p. 42). In the case $0 < \beta \leqslant 1$ and $\omega(\delta, f) = O(\delta^\beta)$ we write $f \in lip\,\beta$, and if $\omega_p(\delta, f) = O(\delta^\beta)$, we write $f \in lip\ (\beta, p)$. The case $\beta > 1$ is of no interest as in this case f turns out to be constant. The class $lip\ (\beta, p)$ with $p = \infty$ will be taken as $lip\ \beta$.

Let

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leqslant K|x-y|^\alpha\},$$

where k is a positive constant, not necessarily same at each occurrence. It is known [7] that H_α is a Banach space with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{x\neq y} \Delta^\alpha f(x,y), \tag{1.9}$$

where

$$\Delta^\alpha f(x,y) = |f(x) - f(y)|\,|x-y|^{-\alpha}, \quad x \neq y$$

and

$$\Delta^0 f(x,y) = 0.$$

The metric induced by the norm (1.5) on H_α is called the Hölder metric. Since

$$\|f\|_\beta \leqslant (2\pi)^{\alpha-\beta} \|f\|_\alpha, \quad 0 \leqslant \beta < \alpha \leqslant 1$$

it follows that $H_\alpha \subseteq H_\beta \subseteq C_{2\pi}$; that is $\{H_\alpha, \|\cdot\|_\alpha\}$ is a family of Banach spaces which decrease as α increases. Hölder metric has been generalized in [2], as follows.

For $0 < \alpha \leqslant 1$,

$$H(\alpha, p) = \{f \in L_p,\quad 0 < p \leqslant \infty : \|f(\cdot+h) - f(\cdot)\|_p \leqslant K|h|^\alpha\}$$

and define for $f \in H(\alpha, \rho)$

$$\|f\|_{(\alpha,p)} = \|f\|_p + \sup_{h\neq 0} \frac{\|f(\cdot+h) - f(\cdot)\|_p}{|h|^\alpha}$$

$$\|f\|_{(0,p)} = \|f\|_p. \tag{1.10}$$

It can be easily verified that (1.10) is a norm for $p \geqslant 1$ and that $H(\alpha, p)$ is a Banach space. Note that $H(\alpha, \infty)$ is the familiar H_α space introduced earlier by Prössdorff [7].

Let $A = (a_{n,k})$ be an infinite matrix that satisfies the following conditions:

$$\|A\| = \sup_n \sum_{k=0}^{\infty} |a_{n,k}| < \infty \tag{1.11}$$

and

$$\sum_{k=0}^{\infty} a_{n,k} = 1 \quad \text{for each} \quad n = 0, 1, 2, \ldots. \tag{1.12}$$

We write $A \in \mathscr{T}$ if conditions (1.11) and (1.12) hold. Let (μ_n) be a positive non-decreasing sequence such that

$$\sum_{k=\mu_n}^{\infty} (k+1)|a_{n,k}| = O(\mu_n). \tag{1.13}$$

Also we need the following additional notations:

$$\Psi(n) = \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}|, \tag{1.14}$$

$$D_n(u) = \frac{\sin(n+1/2)u}{2\sin u/2}, \tag{1.15}$$

$$F_x(u) = \chi_x(u) - \chi_x(\pi/\mu_n), \tag{1.16}$$

$$G(u) = F_x(u) - F_{x+y}(u), \tag{1.17}$$

$$K_n(u) = \sum_{k=0}^{\infty} a_{n,k} D_k(u). \tag{1.18}$$

2. Introduction

By writing Hardy–Littlewood series or in short HL-series we mean the series

$$\sum_{n=1}^{\infty} \frac{s_n(x) - f(x)}{n}. \tag{2.1}$$

We take this opportunity to acknowledge the fact that this nomenclature for the series (2.1) was first given by Mohanty (see [6]).

Hardy and Littlewood [5] have shown that (2.1) is summable $(C, 1)$ to the value

$$\frac{1}{\pi}\int_{0+}^{\pi} \left\{\left(\frac{\pi - u}{2}\right)\cot u/2 - \log(2\sin u/2)\right\}\Phi_x(u)\,du$$

whenever the integral

$$\int_{0+}^{\pi} \Phi_x(u)\frac{1}{2}\cot u/2\,du \tag{2.2}$$

exists, Further ([5], see also [9], p. 122), if

$$\int_0^t |\Phi_x(u)|\,du = o(t) \quad \text{as} \quad t \to o+, \tag{2.3}$$

then (2.1) converges if and only if (2.2) exists. The interest of the HL-series lies in its relationship to the integral (2.2), and these relations are very similar to those between the conjugate series $\sum B_n$ and the integral

$$\int_{0+}^{\pi} \frac{\psi_x(u)}{u}\, du, \tag{2.4}$$

where

$$\psi_x(u) = \frac{1}{2}\{f(x+u) - f(x-u)\}.$$

It is known [9] that if $f \in L$ then (2.4) exists almost everywhere.

On the other hand there exists a continuous function f for which the integral (2.2) diverges for almost all x [5].

At this stage, we remark that the above results on HL-series remain unaltered if we replace HL-series by

$$\frac{1}{2} c_0 + \sum_{n=1}^{\infty} \left\{ \frac{s_n^*(x) - f(x)}{n} \right\}, \tag{2.5}$$

where

$$c_0 = \frac{2}{\pi} \int_0^{\pi} \Phi_x(u) \frac{u}{2} \cot u/2\, du$$

and the series (2.5) is summable $(C, 1)$ to the value (2.2) whenever this integral exists. Thus the convergence or summability problem of (2.5) is the same as that of (2.1), though their sums are different and hence we may term (2.5) as HL-series.

Prössdorff [7] studied the degree of approximation in the Hölder metric and proved the following theorem.

Theorem A [7]. *Let $f \in H_\alpha$ $(0 < \alpha \leqslant 1)$ and $0 \leqslant \beta < \alpha \leqslant 1$. Then*

$$\| \sigma_n(f) - f \|_{(\beta,\infty)} = O(1) \begin{cases} n^{\beta-\alpha}, & (0 < \alpha < 1) \\ n^{\beta-1}(1 + \log n)^{1-\beta}, & \alpha = 1 \end{cases}$$

where $\sigma_n(f)$ is the Fejèr means of the Fourier series of f.

(Remark $\| \sigma_n(f) - f \|_{(\beta,\infty)}$ is in our notation). The case $\beta = 0$ of Theorem A is due to Alexits [1]. With regard to the approximation of functions in L_p norm, the following theorem is due to Quade [8].

Theorem B [8]. *Let $f \in lip\,(\alpha, p)$, $0 < \alpha \leqslant 1$. Then*

$$\| \sigma_n(t) - f \|_{(0,p)} = O(1) \begin{cases} n^{-\alpha} & (p > 1) \\ n^{-\alpha} & (p = 1, 0 < \alpha < 1) \\ (\log n)/n & (p = 1, \alpha = 1) \end{cases}.$$

(Remark $\| \sigma_n(f) - f \|_{(0,p)}$ is in our notation). In a recent paper [2], the degree of approximation in the generalized Hölder metric has been introduced and the following result has been obtained.

Theorem C [2]. *Let $s_n(x)$ be the nth partial sum of* (1.1). *Suppose that $A \in \mathscr{T}$ and let there exist a positive non-decreasing sequence (μ_n) such that* (1.13) *hold. Then for $p \geqslant 1$ and $f \in H(\alpha, p)$, $0 < \alpha \leqslant 1$, $0 \leqslant \beta < \alpha$,*

$$\left\| \sum_{k=0}^{\infty} a_{n,k} s_k - f \right\|_{(\beta,p)}$$

$$= O(1) \begin{cases} (1 + \log(\mu_n/\lambda_n))^{\beta/\alpha} \lambda_n^{\beta-\alpha} + \psi(n)\lambda_n^{1-\alpha+\beta}, & (0 < \alpha \leqslant 1) \\ \dfrac{(1 + \log(\mu_n/\lambda_n))^{\beta}}{\lambda_n^{1-\beta}} + \psi(n)\lambda_n^{\beta}(\log \lambda_n)^{1-\beta}, & (\alpha = 1) \end{cases}$$

where λ_n is any positive non-decreasing sequence such that $\lambda_n \leqslant \mu_n$.

3. Main results

The object of the present paper is to determine the degree of approximation of the series (2.5) by means of A-transform in the generalized Hölder metric. We prove the following theorem.

Theorem. *Suppose that $A \in \mathscr{T}$ and let there exist a positive non-decreasing sequence (μ_n) such that* (1.13) *hold. Let $M_n(x)$ be the A-transform of the series* (2.5). *Then for $p \geqslant 1$ and $f \in H(\alpha, p)$, $0 < \alpha \leqslant 1$, $0 \leqslant \beta < \alpha$*

$$\| M_n(x) - \chi_x(\pi/\mu_n) \|_{(\beta,p)}$$

$$= O(1) \begin{cases} (\log \mu_n)^{\beta} \left[\dfrac{\left(1 + \log\left(\dfrac{\mu_n}{\lambda_n}\right)\right)^{\beta}}{\lambda_n^{1-\beta}} + \lambda_n^{\beta} \psi(n)(\log \lambda_n)^{1-\beta} \right], & \alpha = 1 \\ (\log \mu_n)^{\beta/\alpha} \left[\left(1 + \log \dfrac{\mu_n}{\lambda_n}\right)^{\beta/\alpha} \lambda_n^{\beta-\alpha} + \psi(n)\lambda_n^{1-\alpha+\beta} \right], & 0 < \alpha < 1 \end{cases} \tag{3.1}$$

where $\psi(n)$ is defined in (1.14) *and λ_n is any positive non-decreasing sequence such that $\lambda_n \leqslant \mu_n$.*

Before we take up the proof, we will exhibit below the following.

Fourier character of HL-series (2.5)

Let

$$\chi_x(u) = \int_u^{\pi} \Phi_x(w) \frac{1}{2} \cot w/2 \, dw. \tag{3.2}$$

It is known [3] that χ is even and Lebesgue integrable. Let

$$\chi_x(t) \sim \frac{1}{2} c_0 + \sum_{n=1}^{\infty} c_n \cos nt. \tag{3.3}$$

We have

$$c_0 = \frac{2}{\pi}\int_0^{\pi} \chi_x(t)\,\mathrm{d}t = \frac{2}{\pi}\int_0^{\pi}\left(\int_t^{\pi} \Phi_x(u)\cot u/2\,\mathrm{d}u\right)\mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^{\pi} \Phi_x(u)\frac{1}{2}\cot u/2\,\mathrm{d}u\int_0^{u}\mathrm{d}t,$$

$$= \frac{2}{\pi}\int_0^{\pi} \Phi_x(u)\frac{u}{2}\cot u/2\,\mathrm{d}u \tag{3.4}$$

and for $n \geqslant 1$

$$c_n = \frac{2}{\pi}\int_0^{\pi} \chi_x(t)\cos nt\,\mathrm{d}t = \frac{2}{\pi}\int_0^{\pi}\cos nt\left(\int_t^{\pi} \Phi_x(u)\frac{1}{2}\cot u/2\,\mathrm{d}u\right)\mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^{\pi} \Phi_x(u)\frac{1}{2}\cot u/2\,\mathrm{d}u\int_0^{u}\cos nt\,\mathrm{d}t$$

$$= \frac{2}{\pi n}\int_0^{\pi}\frac{\Phi_x(u)\sin nu\,\mathrm{d}u}{2\tan u/2}$$

$$= \frac{s_n^*(x) - f(x)}{n}. \tag{3.5}$$

Thus, we have the following.

PROPOSITION

The Hardy Littlewood series (2.5) *is Fourier series of even function* $\chi(u)$ *at* $u = 0$.

4. Lemmas

To prove the theorem we use the following Lemmas.

Lemma 1. *If* $\Phi_x(u) \in L_p$, $p \geqslant 1$ *then*

$$\chi(u) \in L_p, \quad p \geqslant 1. \tag{4.1}$$

Proof. The result is proved by Hardy [4] for $p = 1$. For $p > 1$, we choose q with $1/p + 1/q = 1$ and α such that $1/q < \alpha < 1$. We observe that

$$\alpha > \frac{1}{q} = 1 - \frac{1}{p},$$

$$\frac{p}{q} - p\alpha = p\left(1 - \frac{1}{p}\right) - p\alpha = p - p\alpha - 1.$$

Thus we have $0 < p - p\alpha < 1$. Now

$$|\chi_x(t)| \leqslant \int_t^{\pi}\left|\Phi_x(u)\frac{1}{2}\cot u/2\right|\mathrm{d}u$$

$$\leqslant \int_t^{\pi}\frac{|\Phi_x(u)|}{u^{1-\alpha}}\frac{\mathrm{d}u}{u^{\alpha}} \leqslant \left(\int_t^{\pi}\frac{|\Phi_x(u)|^p}{u^{p-p\alpha}}\mathrm{d}u\right)^{1/p}\left(\int_t^{\pi}\frac{\mathrm{d}u}{u^{q\alpha}}\right)^{1/q}$$

$$= \left(\int_t^{\pi} \frac{|\Phi_x(u)|^p}{u^{p-p\alpha}} \, \mathrm{d}u \right)^{1/p} \left[\frac{1}{(q\alpha - 1)} (t^{1-q\alpha} - \pi^{1-q\alpha}) \right]^{1/q}$$

$$\leqslant K \left(\int_t^{\pi} \frac{|\Phi_x(u)|^p \, \mathrm{d}u}{u^{p-p\alpha}} \right)^{1/p} t^{1/q - \alpha}$$

$$\Rightarrow |\chi_x(t)|^p \leqslant K \left(\int_t^{\pi} \frac{|\Phi_x(u)|^p \, \mathrm{d}u}{u^{p-p\alpha}} \right) t^{p/q - p\alpha}$$

$$= K \left(\int_t^{\pi} \frac{|\Phi_x(u)|^p \, \mathrm{d}u}{u^{p-p\alpha}} \right) t^{p-p\alpha-1}.$$

Hence

$$\int_0^{\pi} |\chi_x(t)|^p \, \mathrm{d}t \leqslant K \int_0^{\pi} t^{p-p\alpha-1} \left(\int_t^{\pi} \frac{|\Phi_x(u)|^p}{u^{p-p\alpha}} \, \mathrm{d}u \right) \mathrm{d}t$$

$$= K \int_0^{\pi} \frac{|\Phi_x(u)|^p}{u^{p-p\alpha}} \, \mathrm{d}u \int_o^u t^{p-p\alpha-1} \, \mathrm{d}t$$

$$= K \int_0^{\pi} \frac{|\Phi_x(u)|^p}{u^{p-p\alpha}} \left(\frac{u^{p-p\alpha}}{p-p\alpha} \right) \mathrm{d}u$$

$$= \frac{K}{p-p\alpha} \int_0^{\pi} |\Phi_x(u)|^p \, \mathrm{d}u < \infty.$$

Thus it proves the lemma.

Lemma 2. *Let* $1 \leqslant p \leqslant \infty$. *Then*

(i) $$\| G(u) \|_p = O(1) \begin{cases} u^{\alpha}, & u > \pi/\mu_n \\ \mu_n^{-\alpha}, & u < \pi/\mu_n \end{cases} \tag{4.2}$$

(ii) $$\| G(u) \|_p = O(1) h^{\alpha} \begin{cases} \log \mu_n, & u > \pi/\mu_n \\ \log \dfrac{1}{u}, & u < \pi/\mu_n \end{cases} . \tag{4.3}$$

Proof (i). Since $f \in H(\alpha, p)$ then

$$\| \Phi(w) - \Phi_h(w) \|_p = O(|w|^{\alpha}).$$

From (1.6) we have

$$F_x(u) = \int_u^{\pi/\mu_n} \Phi_x(w) \frac{1}{2} \cot w/2 \, \mathrm{d}w.$$

Now from (1.17) and using generalized Minkowski's inequality for $p \geqslant 1$ we get

$$\| G(u) \|_p = \| F(u) - F_h(u) \|_p = \left(\int_0^{\pi} |F_x(u) - F_{x+h}(u)|^p \, \mathrm{d}x \right)^{1/p}$$

$$= \left\{ \int_0^{\pi} \mathrm{d}x \left| \int_u^{\pi} (\Phi_x(w) - \Phi_{x+h}(w)) \frac{1}{2} \cot w/2 \, \mathrm{d}w \right|^p \right\}^{1/p}$$

$$\leqslant \int_u^\pi \frac{dw}{2\tan w/2} \|\Phi(w)-\Phi_h(w)\|_p$$

$$= O(1)\int_u^\pi w^{\alpha-1}\,dw = O(1)\begin{cases} u^\alpha, & u>\pi/\mu_n \\ \mu_n^{-\alpha}, & u<\pi/\mu_n \end{cases}.$$

Proof (ii). Since $f \in H(\alpha, p)$ then

$$\|\Phi(w)-\Phi_h(w)\| = O(|h|^\alpha).$$

From (1.16), (1.17) and using generalized Minkowski's inequality for $p \geqslant 1$ as above we have

$$\|G(u)\|_p = \|F(u)-F_h(u)\|_p$$

$$\leqslant \int_u^\pi \frac{dw}{2\tan w/2} \|\Phi(w)\Phi_{+h}(w)\|_p$$

$$= O(1)h^\alpha \begin{cases} \log \mu_n, & u>\pi/\mu_n \\ \log \dfrac{1}{u}, & u<\pi/\mu_n \end{cases}.$$

5. Proof of the Theorem

Denoting the nth partial sum of (2.5) by $T_n(x)$ and using (3.6) we have

$$T_n(x) = \frac{1}{2}c_0 + \sum_{k=1}^{n} \frac{S_k^*(x)-f(x)}{k}$$

$$= \frac{1}{2}c_0 + \sum_{k=1}^{n} c_k$$

$$= \frac{2}{\pi}\int_0^\pi \chi_x(u) D_n(x)\,du.$$

We write

$$I_n(x) = M_n(x) - \chi_x(\pi/\mu_n) = \sum_{k=0}^{\infty} a_{n,k} T_k(x) - \chi_x(\pi/\mu_n). \tag{5.1}$$

Now

$$I_n(x) = \frac{2}{\pi}\sum_{k=0}^{\infty} \chi_x(u) \sum_{k=0}^{\infty} a_{n,k} D_k(u)du - \chi_x(\pi/\mu_n)\sum_{k=0}^{\infty} a_{n,k}\frac{2}{\pi}\int_0^\pi D_k(u)\,du$$

$$= \frac{2}{\pi}\int_0^\pi F_x(u)\sum_{k=0}^{\infty} a_{n,k} D_k(u)du$$

$$= \frac{2}{\pi}\int_0^\pi F_x(u) K_n(u) du, \tag{5.2}$$

where $F_x(u)$ and $K_n(u)$ are respectively defined in (1.16) and (1.18).

Note that the change of order of summation and integration is justified provided either side is convergent. We observe that by (1.12) the series for $K_n(u)$ is convergent (even absolutely) and

$$K_n(u) = O(u^{-1})$$

for all $0 < u \leqslant \pi$ and the integral (5.2) exists.

Now by generalized Minkowski's inequality for $p \geqslant 1$, we have

$$\| l_n(x) - l_n(x+y) \|_p \leqslant \frac{2}{\pi} \int_0^{\pi} \| F_x(u) - F_{x+y}(u) \|_p \, |K_n(u)| \, \mathrm{d}u$$

$$= \frac{2}{\pi} \left(\int_0^{\pi/\lambda_n} + \int_{\pi/\lambda_n}^{\pi} \right) \| G(u) \|_p \, |K_n(u)| \, \mathrm{d}u$$

$$= l_1 + l_2, \quad \text{say,} \tag{5.3}$$

$$l_1 = \frac{2}{\pi} \int_0^{\pi/\lambda_n} \| G(u) \|_p \, |K_n(u)| \, \mathrm{d}u$$

$$= \frac{2}{\pi} \left(\int_0^{\pi/\mu_n} + \int_{\pi/\mu_n}^{\pi/\lambda_n} \right) \| G(u) \|_p \, |K_n(u)| \, \mathrm{d}u$$

$$= l_{1,1} + l_{1,2} \quad \text{say.} \tag{5.4}$$

We first note that

$$K_n(u) = \frac{1}{2 \sin u/2} \left(\sum_{k=0}^{\mu_n} + \sum_{k=\mu_n+1}^{\infty} \right) a_{n,k} \sin\left(k + \frac{1}{2} \right) u$$

$$= O(u^{-1}) \left[\sum_{k=0}^{\mu_n} |a_{n,k}| (k+1) u + \sum_{k=\mu_n+1}^{\infty} (k+1) |a_{n,k}| u \right]$$

$$= O(\mu_n) + O(\mu_n) = O(\mu_n). \tag{5.5}$$

By Lemma 2(i) and (5.5) we get

$$l_{1,1} = \frac{2}{\pi} \int_0^{\pi/\mu_n} \| G(u) \|_p \, |K_n(u)| \, \mathrm{d}u$$

$$= O(1) \int_0^{\pi/\mu_n} \mu_n^{1-\alpha} \, \mathrm{d}u$$

$$= O(1) \, \mu_n^{-\alpha}. \tag{5.6}$$

By making use of the fact that

$$\left| \sum_{k=0}^{\infty} a_{n,k} \sin\left(k + \frac{1}{2} \right) u \right| \leqslant \| A \| < \infty \tag{5.7}$$

and by Lemma 2 (i) we have

$$l_{1,2} = \frac{2}{\pi} \int_{\pi/\mu_n}^{\pi/\lambda_n} \| G(u) \|_p \, |K_n(u)| \, \mathrm{d}u$$

$$= O(1) \int_{\pi/\mu_n}^{\pi/\lambda_n} u^{\alpha} u^{-1} \, \mathrm{d}u$$

$$= O(1) \left(\frac{1}{\lambda_n^{\alpha}} \right). \tag{5.8}$$

Combining (5.6) and (5.8) we obtain

$$\begin{aligned} l_1 &= O(1)\mu_n^{-\alpha} + O(1)\frac{1}{\lambda_n^\alpha} \\ &= O(1)\left(\frac{1}{\lambda_n^\alpha}\right). \end{aligned} \tag{5.9}$$

By Abel's transformation

$$\sum_{k=0}^{\infty} a_{n,k} \sin\left(k + \frac{1}{2}\right) u = O(u^{-1}) \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}|. \tag{5.10}$$

Now by (5.10), (1.14) and Lemma 2(i) we get

$$\begin{aligned} l_2 &= \frac{2}{\pi} \int_{\pi/\lambda_n}^{\pi} \| G(u) \|_p |K_n(u)| \, du \\ &= O(1) \int_{\pi/\lambda_n}^{\pi} u^{\alpha-2} \psi(n) du \\ &= O(1)\psi(n) \begin{cases} \log \lambda_n, & \alpha = 1 \\ \lambda_n^{1-\alpha}, & 0 < \alpha < 1 \end{cases}. \end{aligned} \tag{5.11}$$

Again by Lemma 2(ii) and (5.5), we get

$$\begin{aligned} l_{1,1} &= O(1)|y|^\alpha \int_0^{\pi/\mu_n} \log\frac{1}{u} |k_n(u)| \, du \\ &= O(1)|y|^\alpha \mu_n \int_0^{\pi/\mu_n} \log\frac{1}{u} du \\ &= O(1)|y|^\alpha \log \mu_n. \end{aligned} \tag{5.12}$$

By Lemma 2(ii) and (5.7) we obtain

$$\begin{aligned} l_{1,2} &= O(1)|y|^\alpha \log \mu_n \int_{\pi/\mu_n}^{\pi/\lambda_n} |K_n(u)| \, du \\ &= O(1)|y|^\alpha \log \mu_n \int_{\pi/\mu_n}^{\pi/\lambda_n} u^{-1} \, du \\ &= O(1)|y|^\alpha \log \mu_n \log\left(\frac{\mu_n}{\lambda_n}\right). \end{aligned} \tag{5.13}$$

Combining (5.12) and (5.13), we obtain

$$\begin{aligned} l_1 &= O(1)|y|^\alpha \log \mu_n + O(1)|y|^\alpha \log \mu_n \log\left(\frac{\mu_n}{\lambda_n}\right) \\ &= O(1)|y|^\alpha (\log \mu_n)\left(1 + \log\frac{\mu_n}{\lambda_n}\right). \end{aligned} \tag{5.14}$$

Again by Lemma 2(ii), (1.14) and (5.10) we get

$$l_2 = O(1)|y|^\alpha \int_{\pi/\lambda_n}^{\pi} \log\mu_n |K_n(u)|\,\mathrm{d}u$$

$$= O(1)|y|^\alpha \log\mu_n \int_{\pi/\lambda_n}^{\pi} \frac{\psi(n)}{u^2}\,\mathrm{d}u$$

$$= O(1)|y|^\alpha \log\mu_n\, \psi(n)\lambda_n. \tag{5.15}$$

Combining (5.9) and (5.14) we obtain

$$l_1 = l_1^{\beta/\alpha} l_1^{1-\beta/\alpha}$$

$$= O(1)|y|^\beta (\log\mu_n)^{\beta/\alpha}\left(1+\log\frac{\mu_n}{\lambda_n}\right)^{\beta/\alpha}\frac{1}{\lambda_n^{\alpha-\beta}}. \tag{5.16}$$

Again combining (5.11) and (5.15) we get

$$l_2 = l_2^{\beta/\alpha} l_2^{1-\beta/\alpha}$$

$$= \begin{cases} |y|^\beta (\log\mu_n\psi(n)\lambda_n)^{\beta/\alpha}(\psi(n)\log\lambda_n)^{1-\beta/\alpha}, & \alpha = 1 \\ |y|^\beta (\log\mu_n\psi(n)\lambda_n)^{\beta/\alpha}(\psi(n)\lambda_n^{1-\alpha})^{1-\beta/\alpha}, & 0<\alpha<1 \end{cases}$$

$$= O(1)|y|^\beta\psi(n)\begin{cases} \lambda_n^\beta(\log\mu_n)^\beta(\log\lambda_n)^{1-\beta}, & \alpha=1 \\ \lambda_n^{1-\alpha+\beta}(\log\mu_n)^{\beta/\alpha}, & 0<\alpha<1 \end{cases}. \tag{5.17}$$

Hence

$$\sup_{y\neq 0}\frac{\| l_n(x+y) - l_n(x)\|_p}{|y|^\beta}$$

$$= O(1)(\log\mu_n)^{\beta/\alpha}(1+\log(\mu_n/\lambda_n))^{\beta/\alpha}\frac{1}{\lambda_n^{\alpha-\beta}}$$

$$+ O(1)\psi(n)\begin{cases} \lambda_n^\beta(\log\mu_n)^\beta(\log\lambda_n)^{1-\beta}, & \alpha=1 \\ \lambda_n^{1-\alpha+\beta}(\log\mu_n)^{\beta/\alpha}, & 0<\alpha<1. \end{cases} \tag{5.18}$$

It follows from the analysis of the proof of (5.9) and (5.11) that

$$\| l_n(x)\|_p = O(1)\left(\frac{1}{\lambda_n^\alpha}\right) + O(1)\psi(n)\begin{cases} \log\lambda_n, & \alpha=1 \\ \lambda_n^{1-\alpha}, & 0<\alpha<1. \end{cases} \tag{5.19}$$

Now we combine (5.18) and (5.19) to obtain the degree of approximation for $\| l_n(x)\|_{(\beta,p)}$ as

$$\| l_n(x)\|_{(\beta,p)} = O(1)\Bigg[\frac{1}{\lambda_n^\alpha} + \psi(n)\begin{cases} \log\lambda_n, & \alpha=1 \\ \lambda_n^{1-\alpha}, & 0<\alpha<1 \end{cases}$$

$$+ (\log\mu_n)^{\beta/\alpha}(1+\log\mu_n/\lambda_n)^{\beta/\alpha}\frac{1}{\lambda_n^{\alpha-\beta}}$$

$$+ \psi(n)\begin{cases} \lambda_n^\beta(\log\mu_n)^\beta(\log\lambda_n)^{1-\beta}, & \alpha=1 \\ \lambda_n^{1-\alpha+\beta}(\log\mu_n)^{\beta/\alpha}, & 0<\alpha<1 \end{cases}\Bigg].$$

Hence the result follows.

Acknowledgement

The authors are grateful to the referee for useful suggestions.

References

[1] Alexitis G, *Convergence problem on orthogonal series* (New York: Pergamon Press) (1961)
[2] Das G, Ghosh T and Ray B K, Degree of approximation of functions by their Fourier series in the generalized Hölder metric. *Proc. Indian Acad. Sci.* **106** (1996) 139–153
[3] Hardy G H, Notes on some points in the integral calculus (LXVI): The arithmetic mean of Fourier constant, *Messenger Math.* **58** (1928) 50–52
[4] Hardy G H, *Divergent Series* (Oxford: Clarendon Press) (1949)
[5] Hardy G H and Littlewood J E, The allied series of Fourier series, *Proc. London Math. Sci.* **24** (1926) 211–246
[6] Mohanty R, On the absolute convergence of Hardy–Littlewood series. *J. Orissa Math. Soc.* **12–15** (1993–1996) 237–240
[7] Prössdorff S, Zur Konvergenz der Fourier reihen Hölder stetiger Funktionen, *Math. Nachr.* **69** (1975) 7–14
[8] Quade E S, Trigonometric approximation in the mean, *Duke Math. J.* **3** (1937) 529–543
[9] Zygmund A, *Trigonometric Series* (Cambridge University Press, New York) (1968), vol. I

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 121–131.

Fix-points of certain differential polynomials

SUBHAS S BHOOSNURMATH and CHHAYA M HOMBALI*

Department of Mathematics, Karnatak University, Dharwad 580 003, India
*Kittel Science College, Dharwad 580 001, India

MS received 25 February 1997; revised 4 April 1998

Abstract. In this paper, we obtain some results on certain differential polynomials. We use the techniques of Hayman and Yang Lo. The results of Fred Gross on fix-points will be improved and generalized.

Keywords. Nevanlinna theory; meromorphic functions; differential polynomials; fix-points.

1. Introduction

Let $f(z)$ be a transcendental meromorphic function in the finite complex plane C. We assume familiarity with usual notations of Nevanlinna theory as explained in [5].

Let $n_{2j}, n_{3j}, \ldots, n_{kj}$ be non-negative integers. An expression of the form

$$M_j[f] = (f^{(2)})^{n_{2j}}(f^{(3)})^{n_{3j}} \cdots (f^{(k)})^{n_{kj}} \quad 1 \leqslant j \leqslant l$$

is called a monomial in $f^{(2)}$.

Following Sons [6], we call

$$\gamma_{M_j} = n_j = n_{2j} + n_{3j} + \cdots + n_{kj}$$

the degree of the monomial $M_j[f]$ and

$$\Gamma_{M_j} = \Gamma_j = 3n_{2j} + 4n_{3j} + \cdots + (k+1)n_{kj}$$

its weight.

In what follows, we shall be concerned with differential polynomials of the form

$$\Psi(z) = \Psi_f(z) = M_l(f) + \sum_{j=1}^{l-1} a_j(z) M_j(f), \tag{1}$$

where $\gamma_\psi = n =$ degree of

$$\Psi = \max_{1 \leqslant j \leqslant l} \left(\sum_{i=2}^{k} n_{ij} \right) = \max_{1 \leqslant j \leqslant l} n_j$$

and

$$n_1 = \min_{1 \leqslant j \leqslant l} \left(\sum_{i=2}^{k} n_{ij} \right) = \min_{1 \leqslant j \leqslant l} n_j = \min_{1 \leqslant j \leqslant l} \gamma_{M_j}$$

and the weight of $\Psi = \Gamma_\Psi = 3n_{2l} + 4n_{3l} + \cdots + (k+1)n_{kl}$. The coefficients $a_j(z)$ are meromorphic functions satisfying $T(r, a_j(z)) = S(r,f)$, $1 \leqslant j \leqslant l-1$.

If $n = n_j = \gamma_\Psi$ for $1 \leqslant j \leqslant l$, then $\Psi(z)$ is called a homogeneous differential polynomial in $f^{(2)}$ of degree n. $z_0 \in C$ is called a fix-point of $f(z)$ if $f(z_0) = z_0$.

In [4], Gross Fred proved the following theorem with a restriction on the number of poles of $f(z)$.

Theorem A. *Let $f(z)$ be a transcendental meromorphic function with at most a finite number of poles. Let*

$$\Psi_f(z) = \sum_{j=0}^{l} a_j(z) f^{(j)}(z),$$

where $T(r, a_j(z)) = S(r, f), 0 \leqslant j \leqslant l, \Psi_f(z)$ is not identically constant, $\Psi_z \not\equiv z$. Then one of the functions $\Psi_f(z)$, $f(z)$ has infinitely many fix-points.

In this paper, we suppress the hypothesis on the number of poles of $f(z)$ and obtain similar interesting results on certain differential polynomials of the form (1).

We shall be using the techniques developed by Hayman [5] and the analysis of Yang Lo [7] in the course of this paper.

Our main result is the following theorem.

Theorem 1. *Suppose $f(z)$ is a transcendental meromorphic function in C, $f(0) \neq 0, \neq \infty$, $\Psi(z)$ is a differential polynomial of the form* (1) *and $\Psi(z)$ is not identically constant. Let*

$$\Gamma_\Psi \geqslant \Gamma_{M_j} + 3, \qquad 1 \leqslant j \leqslant l-1 \tag{2}$$

and

$$3n_1 > 2n. \tag{3}$$

Then either $f(z)$ has infinitely many fix-points or $\Psi(z)$ has infinitely many fix-points.

For the proof of this theorem, we need the following lemmas.

Lemma 1. [3] *If $P(f)$ is a homogeneous differential polynomial in f of degree n, then*

$$m\left(r, \frac{P(f)}{f^n}\right) = S(r, f).$$

Lemma 2. [2] *If $P(f)$ is a differential polynomial in f of degree n and weight Γ_P, then*

$$T(r, P(f)) \leqslant \Gamma_P T(r, f) + S(r, f).$$

Therefore

$$S(r, P(f)) = S(r, f).$$

As a consequence of this we claim that $S(r, (\psi/f^n)) = S(r, f)$ where ψ is as given in (1). Now,

$$\begin{aligned} T\left(r, \frac{\psi}{f^n}\right) &\leqslant T(r, \psi) + nT\left(r, \frac{1}{f}\right) \\ &\leqslant \Gamma_\psi T(r, f) + nT(r, f) + S(r, f) \\ &\leqslant O(T(r, f)) + S(r, f) \end{aligned} \tag{4}$$

In view of (4),

$$S\left(r, \frac{\psi}{f^n}\right) = S(r, f).$$

Lemma 3. *Suppose* $f(z)$ *and* $\psi(z)$ *are as given in Theorem 1. Then*

(a) $$m\left(r,\frac{\Psi}{f^n}\right)\leqslant(n-n_1)m\left(r,\frac{1}{f}\right)+S(r,f),$$

(b) $$m\left(r,\frac{\Psi'}{f^n}\right)\leqslant(n-n_1)m\left(r,\frac{1}{f}\right)+S(r,f).$$

Proof. (a) For the proof of this see [1].
(b) Since

$$\frac{\psi'}{f^n}=\left(\frac{\psi}{f^n}\right)'+n\frac{f'}{f^n}\frac{\psi}{f^n}$$
$$=\frac{\psi}{f^n}\left\{n\frac{f'}{f}+\frac{(\psi/f^n)'}{\dfrac{\psi}{f^n}}\right\},$$

We obtain, by using Lemma 2.

$$m\left(r,\frac{\psi'}{f^n}\right)\leqslant m\left(r,\frac{\psi}{f^n}\right)+S(r,f)+m\left(r,\frac{(\psi/f^n)'}{\dfrac{\psi}{f^n}}\right)$$
$$=m\left(\left(r,\frac{\psi}{f^n}\right)+S(r,f)+S\left(\left(r,\frac{\psi}{f^n}\right)\right.\right.$$
$$=m\left(r,\frac{\psi}{f^n}\right)+S(r,f)$$
$$\leqslant(n-n_1)m\left(r,\frac{1}{f}\right)+S(r,f).$$

Lemma 4. *Let* $f(z)$, and $\Psi(z)$ *be as in Theorem 1. If* $\Psi(0)\neq 0$ *then*

$$\alpha T(r,f)<\bar{N}(r,f)+\alpha N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{\Psi-z}\right)-N\left(r,\frac{1}{z\Psi'-\Psi}\right)+S(r,f), \tag{5}$$

where $\alpha=3n_1-2n$.

Proof. Consider the identify

$$\frac{1}{fn}=\frac{\Psi}{zf^n}-\frac{z\Psi'-\Psi}{zf^n}\cdot\frac{\Psi-z}{z\Psi'-\Psi}.$$

This leads to

$$m\left(r,\frac{1}{f^n}\right)\leqslant m\left(r,\frac{\Psi}{zf^n}\right)+m\left(r,\frac{z\Psi'-\Psi}{zf^n}\right)+m\left(r,\frac{\Psi-z}{z\Psi'-\Psi}\right)+\log 2. \tag{6}$$

Applying the Jensen-Nevanlinna formula to $m(r,(1/f^n))$ and $m(r,((\Psi-z))/(z\Psi'-\Psi))$, we get

$$m\left(r,\frac{1}{f^n}\right)=nT(r,f)-N\left(r,\frac{1}{f^n}\right)+\log\frac{1}{|f^n(0)|}, \tag{7}$$

$$m\left(r,\frac{\Psi-z}{z\Psi'-\Psi}\right)=m\left(r,\frac{z\Psi'-\Psi}{\Psi-z}\right)+N\left(r,\frac{z\Psi'-\Psi}{\Psi-z}\right)-N\left(r,\frac{\Psi-z}{z\Psi'-\Psi}\right)$$
$$=m\left(r,\frac{z\Psi'-\Psi}{\Psi-z}\right)+N\left(r,\frac{1}{\Psi-z}\right)-N(r,\Psi-z)$$
$$+N(r,z\Psi'-\Psi)-N\left(r,\frac{1}{z\Psi'-\Psi}\right)\leqslant m\left(r,\frac{z\Psi'-\Psi}{\Psi-z}\right)$$
$$+N\left(r,\frac{1}{\Psi-z}\right)+\bar{N}(r,f)-N\left(r,\frac{1}{z\Psi'-\Psi}\right). \tag{8}$$

Substituting (8), (7) in (6), we have

$$nT(r,f)<nN\left(r,\frac{1}{f}\right)+N\log|f(0)|+m\left(r,\frac{\Psi}{zf^n}\right)$$
$$+m\left(r,\frac{z\Psi'-\Psi}{\Psi-z}\right)+m\left(r,\frac{z\Psi'-\Psi}{zf^n}\right)+\bar{N}(r,f)$$
$$+N\left(r,\frac{1}{\Psi-z}\right)-N\left(r,\frac{1}{z\Psi'-\Psi}\right)+\log 2. \tag{9}$$

Since by Lemma 2,

$$m\left(r,\frac{(\Psi-z)'}{\Psi-z}\right)=S(r,\Psi)=S(r,f)$$

and using Lemma 3, we have,

$$m\left(r,\frac{\Psi}{zf^n}\right)+m\left(r,\frac{z\Psi'-\Psi}{zf^n}\right)+m\left(r,\frac{z\Psi'-\Psi}{\Psi-z}\right)$$
$$\leqslant m\left(r,\frac{1}{z}\right)+m\left(r,\frac{\Psi}{f^n}\right)+m\left(r,\frac{\Psi'}{f^n}\right)+m\left(r,\frac{1}{z}\right)$$
$$+m\left(r,\frac{\Psi}{f^n}\right)+m(r,z)+m\left(r,\frac{(\Psi-z)'}{\Psi-z}\right)+2\log 2$$
$$\leqslant 2m\left(r,\frac{\Psi}{f^n}\right)+m\left(r,\frac{\Psi'}{f^n}\right)+S(r,f)$$
$$\leqslant 3(n-n_1)m\left(r,\frac{1}{f}\right)+S(r,f)$$
$$\leqslant 3\,(n-n_1)\,T(r,f)-3(n-n_1)N\left(r,\frac{1}{f}\right)+S(r,f). \tag{10}$$

Substituting (10) in (9) we get

$$(3n_1-2n)\,T(r,f)<\bar{N}(r,f)+(3n_1-2n)N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{\Psi-z}\right)$$
$$-N\left(r,\frac{1}{z\Psi'-\Psi}\right)+S(r,f).$$

Lemma 5. *Let $f(z)$ and $\Psi(z)$ be as given in Lemma* 4, *then*

$$2\bar{N}_{1)}(r,f)\leqslant \bar{N}(r,f)+\bar{N}\left(r,\frac{1}{\Psi-z}\right)+\bar{N}\left(r,\frac{1}{z\Psi'-\Psi}\right)$$
$$+m\left(r,\frac{g'}{g}\right)+\log|C_\lambda|+2\log^+ r, \tag{11}$$

where

$$g(z)=\frac{(z\Psi'-\Psi)^{\Gamma}\Psi}{z^{\Gamma}\Psi^{+1}(z-\Psi)^{\Gamma}\Psi^{+1}} \tag{12}$$

and C_λ is the first non zero term of Taylor series of $g(z)/g'(z)$ in the neighbourhood of origin, and $\bar{N}_{1)}(r,f)$ denotes the number of simple poles of $f(z)$.

Proof. Let z_0 be a simple pole of $f(z)$ and $z_0\neq 0$. Then $f(z)=(a/z-z_0)+O(1)$, $a\neq 0$ is the neighbourhood of z_0,

$$f'(z)=\frac{(-1)a}{(z-z_0)^2}+O(1)$$

$$f^{(2)}(z)=\frac{(-1)^2 2!a}{(z-z_0)^3}+O(1)$$

$$f^{(k)}(z)=\frac{(-1)^k k!a}{(z-z_0)^{k+1}}+O(1)$$

$$M_j[f]=(f^{(2)})^{n_{2j}}(f^{(3)})^{n_{3j}}\cdots(f^{(k)})^{n_{kj}}\quad 1<j\leqslant l.$$

Now

$$(f^{(2)})^{n_{2j}}=\left(\frac{(-1)^2 2!a}{(z-z_0)^3}+O(1)\right)^{n_{2j}}=\frac{(-1)^{2n_{2j}}a^{n_{2j}}(2!)^{n_{2j}}}{(z-z_0)^{3n_{2j}}}+O\left(\frac{1}{(z-z_0)^{3n}2j^{-3}}\right),$$

$$(f^{(3)})^{n_{3j}}=\frac{(-1)^{3n_{3j}}(3!)^{n_{3j}}a^{n_{3j}}}{(z-z_0)^{(4)n_{3j}}}+O\left(\frac{1}{(z-z_0)^{4n_{3j}-4}}\right),$$

$$(f^{(k)})^{n_{kj}}=\frac{(-1)^{kn_{kj}}(k!)^{n_{kj}}a^{n_{kj}}}{(z-z_0)^{(k+1)n_{kj}}}+O\left(\frac{1}{(z-z_0)^{(k+1)n_{kj}-(k+1)}}\right).$$

Therefore

$$M_j[f]=\frac{(-1)^{\Gamma_{M_j}-n_j}a^{n_j}(2!)^{n_{2j}}(3!)^{n_{3j}}\cdots(k!)^{n_{kj}}}{(z-z_0)^{\Gamma_{M_j}}}+O\left(\frac{1}{(z-z_0)^{\Gamma_{M_j}-3}}\right).$$

Since $\Gamma_\Psi\geqslant\Gamma_{M_j}+3$ for $1\leqslant j\leqslant l-1$,

$$\Psi(z)=\frac{(-1)^{\Gamma_\Psi-n_l}a^{n_l}\beta}{(z-z_0)^{\Gamma_\Psi}}+O\left(\frac{1}{(z-z_0)^{\Gamma_\Psi-3}}\right),$$

where $\beta=(2!)^{n_{2l}}(3!)^{n_{3l}}\cdots(k!)^{n_{kl}}$ and

$$\Psi'(z)=\frac{(-1)^{\Gamma_\Psi-n_l+1}a^{n_l}\beta\Gamma_\Psi}{(z-z_0)^{\Gamma_\Psi+1}}+O\left(\frac{1}{(z-z_0)^{\Gamma_\Psi-2}}\right).$$

We have

$$g(z)=\frac{(z\Psi'-\Psi)^{\Gamma}\Psi}{z^{\Gamma_\Psi+1}(z-\Psi)^{\Gamma_\Psi+1}}=\frac{N}{D}.$$

Now

$$\begin{aligned}
N &= (z\Psi' - \Psi)^{\Gamma_\Psi}\\
&= z^{\Gamma}\Psi(\Psi')^{\Gamma}\Psi + \Gamma_\Psi z^{\Gamma_\Psi - 1}(\Psi')^{\Gamma_\Psi + 1}(-\Psi) + \Gamma_\Psi c_2 z^{\Gamma_\Psi - 2}(\Psi')^{\Gamma_\Psi - 2}(-\Psi)^2 + \ldots\\
&= \frac{z^{\Gamma_\Psi}(-1)^{(\Gamma_\Psi - n_l + 1)\Gamma_\Psi}\beta^{\Gamma_\Psi}a^{n_l\Gamma_\Psi}(T_\psi)^{\Gamma_\Psi}}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)}}\\
&\quad + \frac{\Gamma_\Psi z^{\Gamma_\Psi - 1}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi - 1)}\beta^{\Gamma_\Psi - 1}a^{n_l(\Gamma_\Psi - 1)}(T_\Psi)^{\Gamma_\Psi - 1}}{(z-z_0)^{(\Gamma_\Psi + 1)(\Gamma_\Psi - 1)}}\\
&\quad \times \frac{(-1)^{\Gamma_\Psi - n_l + 1}\beta a^{n_l}}{(z-z_0)^{\Gamma_\Psi}} + O\left(\frac{1}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)-2}}\right).\\
&= \frac{z^{\Gamma_\Psi}(-1)^{(\Gamma_\Psi - n_l + 1)\Gamma_\Psi}\beta^{\Gamma_\Psi}a^{n_l\Gamma_\Psi}(T_\psi)^{\Gamma_\Psi}}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)}}\\
&\quad + \frac{z^{\Gamma_\Psi - 1}(-1)^{(\Gamma_\Psi - n_l + 1)\Gamma_\Psi}\beta^{\Gamma_\Psi}a^{n_l\Gamma_\Psi}(\Gamma_\Psi)^{\Gamma_\Psi}}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)-1}} + O\left(\frac{1}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)-2}}\right)\\
&= \frac{(z_0 + z - z_0)^{\Gamma_\Psi}(-1)^{(\Gamma_\Psi - n_l + 1)\Gamma_\Psi}\beta^{\Gamma_\Psi}a^{n_l\Gamma_\Psi}(T_\Psi)^{\Gamma_\Psi}}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)}}\\
&= \frac{(z_0 + z - z_0)^{\Gamma_\Psi - 1}(-1)^{(\Gamma_\Psi - n_l + 1)\Gamma_\Psi}\beta^{\Gamma_\Psi}a^{n_l\Gamma_\Psi}(T_\Psi)^{\Gamma_\Psi}}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)-1}} + O\left(\frac{1}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)-2}}\right)\\
&= \frac{{z_0}^{\Gamma_\Psi}(-1)^{(\Gamma_\Psi - n_l + 1)\Gamma_\Psi}\beta^{\Gamma_\Psi}a^{n_l\Gamma_\Psi}(T_\Psi)^{\Gamma_\Psi}}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)}}\\
&\quad + \frac{{z_0}^{\Gamma_\Psi - 1}(-1)^{(\Gamma_\Psi - n_l + 1)\Gamma_\Psi}\beta^{\Gamma_\Psi}a^{n_l\Gamma_\Psi}(T_\Psi)^{\Gamma_\Psi}(\Gamma_\Psi + 1)}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)-1}} + O\left(\frac{1}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)-2}}\right)
\end{aligned}$$

and

$$\begin{aligned}
D &= z^{\Gamma_\Psi + 1}(z - \Psi)^{\Gamma_\Psi + 1}\\
&= (-1)^{\Gamma_\Psi + 1}(z\Psi - z^2)^{\Gamma_\Psi + 1}\\
&= (-1)^{\Gamma_\Psi + 1}\{z^{\Gamma_\Psi + 1}\Psi^{\Gamma_\Psi + 1} + (\Gamma_\Psi + 1)z^{\Gamma_\Psi}\Psi^{\Gamma_\Psi}(-z)^2 + \cdots\}\\
&= (-1)^{\Gamma_\Psi + 1}z^{\Gamma_\Psi + 1}\Psi^{\Gamma_\Psi + 1} + (\Gamma_\Psi + 1)z^{\Gamma_\Psi + 2}(-1)^{\Gamma_\Psi}(\Psi)^{\Gamma_\Psi} + \cdots.
\end{aligned}$$

Now

$$\begin{aligned}
(\Psi(z))^{\Gamma_\Psi + 1} &= \left(\frac{(-1)^{\Gamma_\Psi - n_l}\beta a^{n_l}}{(z-z_0)^{\Gamma_\Psi}} + O\left(\frac{1}{(z-z_0)^{\Gamma_\Psi - 3}}\right)\right)^{\Gamma_\Psi + 1}\\
&= \frac{(-1)^{(\Gamma_\Psi - n_l)(\Gamma_\Psi + 1)}\beta^{(\Gamma_\Psi + 1)}a^{n_l(\Gamma_\Psi + 1)}}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)}} + O\left(\frac{1}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)-3}}\right)\\
(\Psi(z))^{\Gamma_\Psi} &= \frac{(-1)^{(\Gamma_\Psi - n_l)\Gamma_\Psi}\beta^{\Gamma_\Psi}a^{n_l T_\Psi}}{(z-z_0)^{\Gamma_\Psi \times \Gamma_\Psi}} + O\left(\frac{1}{(z-z_0)^{\Gamma_\Psi \times \Gamma_\Psi - 3}}\right).
\end{aligned}$$

Since, $\Gamma_\Psi \geqslant \Gamma_{M_j} + 3$, $1 \leqslant j \leqslant -1$, we can write

$$D = \frac{z^{\Gamma_\Psi + 1}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi + 1)}\beta^{\Gamma_\Psi + 1}a^{n_l(\Gamma_\Psi + 1)}}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)}} + O\left(\frac{1}{(z-z_0)^{\Gamma_\Psi(\Gamma_\Psi+1)-3}}\right)$$

$$= \frac{(z_0 + z - z_0)^{\Gamma_\Psi + 1}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi + 1)}\beta^{\Gamma_\Psi + 1}a^{n_l(\Gamma_\Psi + 1)}}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1)}}$$

$$+ O\left(\frac{1}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1) - 3}}\right)$$

$$= \frac{z_0{}^{\Gamma_\Psi + 1}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi + 1)}\beta^{\Gamma_\Psi + 1}a^{n_l(\Gamma_\Psi + 1)}}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1)}}$$

$$+ \frac{(\Gamma_\Psi + 1)z_0{}^{\Gamma_\Psi}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi + 1)}\beta^{\Gamma_\Psi + 1}a^{n_l(\Gamma_\Psi + 1)}}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1) - 1}}$$

$$+ \frac{(\Gamma_\Psi + 1)c_2 z_0{}^{\Gamma_\Psi - 1}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi + 1)}\beta^{\Gamma_\Psi + 1}a^{n_l(\Gamma_\Psi + 1)}}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1) - 2}}$$

$$+ O\left(\frac{1}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1) - 3}}\right)$$

$$= \frac{z_0{}^{\Gamma_\Psi - 1}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi + 1)}\beta^{\Gamma_\Psi + 1}a^{n_l(\Gamma_\Psi + 1)}}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1)}}$$

$$+ \frac{(\Gamma_\Psi + 1)z_0{}^{\Gamma_\Psi}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi + 1)}\beta^{\Gamma_\Psi + 1}a^{n_l(\Gamma_\Psi + 1)}}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1) - 1}} + O\left(\frac{1}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1) - 2}}\right).$$

Hence

$$g(z) = \frac{N}{D} = \frac{(\Gamma_\Psi)^{\Gamma_\Psi}}{z_0 a^{n_l}(-1)^{\Gamma_\Psi - n_l + 1}}(1 + 0(z - z_0)^2).$$

Therefore, $g(z_0) \neq 0$, $\neq \infty$ in a neighbourhood of z_0 and $g'(z_0) = 0$. Thus,

$$\bar{N}_{1)}(r, f) \leqslant N_0\left(r, \frac{1}{g'}\right) + \log^+ r, \tag{13}$$

where $N_0(r, (1/g'))$ denotes the counting function of zeros of $g'(z)$ which are not zeros $g(z)$. Since

$$g(z) = \frac{(z\Psi' - \Psi)^{\Gamma_\Psi}}{z^{\Gamma_\Psi + 1}(z - \Psi)^{\Gamma_\Psi + 1}}$$

$$\log g(z) = \Gamma_\Psi \log(z\Psi' - \Psi) - (\Gamma_\Psi + 1)\log z - (\Gamma_\Psi + 1)\log(z - \Psi),$$

$$\frac{g'(z)}{g(z)} = \frac{\Gamma_\Psi z\Psi''}{z\Psi' - \Psi} - \frac{\Gamma_\Psi + 1}{z} - \frac{(\Gamma_\Psi + 1)(\Psi' - 1)}{\Psi - z},$$

$$\frac{g'(z)}{g(z)} = \frac{\Gamma_\Psi z^2\Psi''(\Psi - z) - (\Gamma_\Psi + 1)(z\Psi' - \Psi)(\Psi - z) - (\Gamma_\Psi + 1)(\Psi' - 1)(z\Psi' - \Psi)z}{z(\Psi - z)(z\Psi' - \Psi)}$$

$$\frac{g(z)}{g'(z)} = \frac{z(\Psi - z)(z\Psi' - \Psi)}{\Gamma_\Psi z^2\Psi''(\Psi - z) - (\Gamma_\Psi + 1)(z\Psi' - \Psi)(\Psi - z) - (\Gamma_\Psi + 1)(\Psi' - 1)(z\Psi' - \Psi)z}$$

$$= \frac{-1}{\Gamma_\Psi + 1}z + O(z^2)$$

is the Taylor series of $g(z)/g'(z)$ in the neighbourhood of the origin.

If C_λ is the first nonzero term of Taylor series of $g(z)/g'(z)$ in the neighbourhood of origin, then

$$\begin{aligned}\log|C_\lambda| &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{g(re^{i\phi})}{g'(re^{i\phi})}\right|d\phi + N\left(r,\frac{g}{g'}\right) - N\left(r,\frac{g'}{g}\right)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{g(re^{i\phi})}{g'(re^{i\phi})}\right|d\phi - \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{g'(re^{i\phi})}{g(re^{i\phi})}\right|d\phi\\ &\quad + N(r,g) - N(r,g') + N\left(r,\frac{1}{g'}\right) - N\left(r,\frac{1}{g}\right)\\ &= m\left(r,\frac{g}{g'}\right) - m\left(r,\frac{g'}{g}\right) - \bar{N}(r,g) + N_0\left(r,\frac{1}{g'}\right) - \bar{N}\left(r,\frac{1}{g}\right).\end{aligned}$$

Therefore

$$N_0\left(r,\frac{1}{g'}\right) \leqslant \bar{N}(r,g) + \bar{N}\left(r,\frac{1}{g}\right) + m\left(r,\frac{g'}{g}\right) + \log|C_\lambda|. \tag{14}$$

From the expression of $g(z)$ it is clear that any zero or pole of $g(z)$ can only occur at zeros of $\Psi(z) - z$, $z = 0$ and poles of $f(z)$ of order greater than one and zeros of $z\Psi' - \Psi$. Therefore

$$\bar{N}(r,g) + \bar{N}\left(r,\frac{1}{g}\right) < \bar{N}_{(2}(r,f) + \bar{N}\left(r,\frac{1}{\Psi - z}\right) + \bar{N}\left(r,\frac{1}{z\Psi' - \Psi}\right) + \log^+ r, \tag{15}$$

where $\bar{N}_{(2}(r,f)$ denotes the counting function of multiple poles of $f(z)$, each of them counted only once.

Since

$$\bar{N}_{(2}(r,f) = \bar{N}(r,f) - \bar{N}_{1)}(r,f),$$

We get

$$\begin{aligned}2\bar{N}_{1)}(r,f) \leqslant \bar{N}(r,f) + \bar{N}\left(r,\frac{1}{\Psi - z}\right) + \bar{N}\left(r,\frac{1}{z\Psi' - \Psi}\right)\\ + m\left(r,\frac{g'}{g}\right) + \log|C_\lambda| + 2\log^+ r\end{aligned}$$

using (13), (14) and (15).

Lemma 6. *Let $f(z)$ and $\Psi(z)$ be as in Lemma* 4. *Then*

$$\begin{aligned}T(r,f) < \frac{3\alpha}{3\alpha - 2}N\left(r,\frac{1}{f}\right) + \frac{4}{3\alpha - 2}N\left(r,\frac{1}{\Psi - z}\right)\\ - \frac{2\alpha}{3\alpha - 2}N\left(r,\frac{1}{z\Psi' - \Psi}\right) + S(r,f),\end{aligned} \tag{16}$$

where $\alpha = 3n_1 - 2n$.

Proof. We have, be using Lemmas 4 and 5,

$$
\begin{aligned}
2\bar{N}(r,f) &\leqslant N(r,f)+\bar{N}_{(1)}(r,f)\\
&< T(r,f)+\bar{N}_{1)}(r,f)\\
&< \frac{1}{\alpha}\bar{N}(r,f)+N\left(r,\frac{1}{f}\right)+\frac{1}{\alpha}N\left(r,\frac{1}{\psi-z}\right)-\frac{1}{\alpha}N\left(r,\frac{1}{z\Psi-\Psi}\right)\\
&\quad+\bar{N}_{(1)}(r,f)+S(r,f)\\
&< \frac{1}{\alpha}\bar{N}(r,f)+N\left(r,\frac{1}{f}\right)+\frac{1}{\alpha}N\left(r,\frac{1}{\psi-z}\right)-\frac{1}{\alpha}N\left(r,\frac{1}{z\Psi-\Psi}\right)+\frac{1}{2}\bar{N}(r,f)\\
&\quad+\frac{1}{2}\bar{N}\left(r,\frac{1}{\psi-z}\right)+\frac{1}{2}N\left(r,\frac{1}{z\Psi'-\Psi}\right)+\frac{1}{2}m\left(r,\frac{g'}{g}\right)+\frac{1}{2}\log|C_\lambda|\\
&\quad+\log^+ r+S(r,f),
\end{aligned}
$$

$$
\begin{aligned}
\left(2-\frac{2+\alpha}{2\alpha}\right)\bar{N}(r,f) &< N\left(r,\frac{1}{f}\right)+\frac{2+\alpha}{2\alpha}N\left(r,\frac{1}{\Psi-z}\right)\\
&\quad-\left(\frac{2-\alpha}{2\alpha}\right)N\left(r,\frac{1}{z\Psi'-\Psi}\right)+\frac{1}{2}m\left(r,\frac{g'}{g}\right)+S(r,f),
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{3\alpha-2}{2\alpha}\bar{N}(r,f) < N\left(r,\frac{1}{f}\right)+\frac{2+\alpha}{2\alpha}N\left(r,\frac{1}{\Psi-z}\right)-\left(\frac{2-\alpha}{2\alpha}\right)N\left(r,\frac{1}{z\Psi'-\Psi}\right)\\
+\frac{1}{2}m\left(r,\frac{g'}{g}\right)+S(r,f).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\bar{N}(r,f) < \frac{2\alpha}{3\alpha-2}N\left(r,\frac{1}{f}\right)+\frac{2+\alpha}{3\alpha-2}N\left(r,\frac{1}{\Psi+z}\right)\\
-\frac{2-\alpha}{3\alpha-2}N\left(r,\frac{1}{z\Psi'-\Psi}\right)+S(r,f).
\end{aligned}
\tag{17}
$$

Since

$$
\frac{g'}{g}=\frac{\Gamma_\Psi z\Psi^{\cdot\cdot}}{z\Psi'-\Psi}-\frac{\Gamma_\Psi+1}{z}-\frac{(\Gamma_\Psi+1)(\Psi'-1)}{\Psi-z}
$$

and, by Lemma 2,

$$
\begin{aligned}
m\left(r,\frac{g'}{g}\right) \leqslant m\left(r,\frac{(z\Psi'-\Psi)'}{z\Psi'-\Psi}\right)+m\left(r,\frac{1}{z}\right)+m\left(r,\frac{(\Psi-z)'}{\Psi-z}\right)\\
+\log 3+2\log(\Gamma_\Psi+1)+\log\Gamma_\Psi=S(r,f).
\end{aligned}
$$

Substituting (17) in (5), we get

$$
\begin{aligned}
\alpha T(r,f) < \frac{2\alpha}{3\alpha-2}N\left(r,\frac{1}{f}\right)+\frac{2+\alpha}{3\alpha-2}N\left(r,\frac{1}{\Psi-z}\right)-\frac{2-\alpha}{3\alpha-2}N\left(r,\frac{1}{z\Psi'-\Psi}\right)\\
+\alpha N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{\Psi-z}\right)-N\left(r,\frac{1}{z\Psi'-\Psi}\right)+S(r,f)
\end{aligned}
$$

$$< \frac{3\alpha^2}{3\alpha-2} N\left(r, \frac{1}{f}\right) + \frac{4\alpha}{3\alpha-2} N\left(r, \frac{1}{\Psi - z}\right) - \frac{2\alpha}{3\alpha-2} N\left(r, \frac{1}{z\Psi' - \Psi}\right) + S(r,f),$$

$$T(r,f) < \frac{3\alpha}{3\alpha-2} N\left(r, \frac{1}{f}\right) + \frac{4}{3\alpha-2} N\left(r, \frac{1}{\Psi - z}\right)$$

$$- \frac{2}{3\alpha-2} N\left(r, \frac{1}{z\Psi' - \Psi}\right) + S(r,f). \tag{18}$$

Proof of Theorem 1. From (18), we have

$$T(r,f) < \frac{3\alpha}{3\alpha-2} N\left(r, \frac{1}{f}\right) + \frac{4}{3\alpha-2} N\left(r, \frac{1}{\Psi - z}\right)$$

$$- \frac{2}{3\alpha-2} N\left(r, \frac{1}{z\Psi' - \Psi}\right) + S(r,f).$$

Changing from f to $f - z$, we have

$$\Psi_{f-z}(z) = M_l[f - z] + \sum_{j=1}^{l-1} a_j(z) M_j[f - z]$$

$$= \Psi_f(z) = \Psi(z)$$

and noting that

$$T(r,f) = T(r, f - z + z) \leqslant T(r, f - z) + T(r,z) \leqslant T(r, f - z) + O(\log r),$$

we get

$$T(r,f) < \frac{3\alpha}{3\alpha-2} N\left(r, \frac{1}{f - z}\right) + \frac{4}{3\alpha-2} N\left(r, \frac{1}{\Psi - z}\right) + S(r,f).$$

If the equations $f(z) = z$ and $\Psi(z) = z$ have only finite number of roots, we deduce

$$(1 + O(1))T(r,f) = O(\log r) \text{ as } r \to \omega$$

so that $f(z)$ is rational, which contradicts our hypothesis. This proves the theorem.

Remark. If $\Psi(z)$ is a homogeneous differential polynomial in $f^{(2)}$ of degree n, then $n = \alpha = n_1$.

Acknowledgement

The authors wish to thank the referee for his comments and careful observations. Lemma 3 was proved in the present improved form following the guidelines of the referee.

References

[1] Chuang Chi-tai, *On differential polynomials, Analysis of one complex* variables (Singapore 1987).
[2] Doeringer W, Exceptional values of differential polynomials. *Pacific J. Math.* **98** (1982) 55–62
[3] Gopalakrishna H S and Subhas S Bhoosnurmath, On the deficiencies of differential polynomials, *The Karnatak Univ. J. Science* **XVIII** (1973) 329–335

[4] Gross Fred, On fix-points of meromorphic functions, *Math. Scand.* **36** (1968) 230–231
[5] Hayman W K, *Meromorphic functions* (Oxford University Press) (1964)
[6] Sons L R, Deficiencies of monomials, *Math Z.* **III** (1969) 53–68
[7] Yang Lo, Normal families and fix-points of meromorphic functions, *Indiana Univ. Math. J.* **35** (1986) 171–191

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 133–136.

On the hopficity of the polynomial rings

S P TRIPATHI

Department of Mathematics, K. M. College, University of Delhi, Delhi 110007, India

MS received 19 January 1998

Abstract. In this paper we prove that if a ring R satisfies the condition that for some integer $n > 1, a^n = a$ for every a in R, then R a hopfian ring implies that the ring $R[T]$ of polynomials is also hopfian. This generalizes a recent result of Varadarajan which states that if R is a Boolean hopfian ring then the ring $R[T]$ is also hopfian. We show furthermore that there are numerous rings R satisfying the hypothesis of our theorem which are neither Boolean nor Noetherian.

Keywords. Hopfian rings.

Let R be a ring with $1 \neq 0$ and suppose $R[T]$ is the polynomial ring over R in one indeterminate T. Recall that a ring R is said to be hopfian (resp. cohopfian) if every onto (resp. one-one) ring homomorphism $f: R \to R$ is automatically an isomorphism (see [2] and [4] for details). The question whether or not R hopfian implies $R[T]$ hopfian is still unresolved. A Noetherian ring is easily seen to be hopfian (see [1], p. 78, exer. 1). By the Hilbert basis theorem R Noetherian implies the ring $R[T]$ is Noetherian and hence is hopfian. Therefore the general question about the hopficity of the ring $R[T]$ reduces to those rings R which are hopfian but not Noetherian. Using an interesting existence result proved in [2], Varadarajan has already pointed out that there exist such rings in abundance [5]. He has also proved the following result: If R is a Boolean hopfian ring then the ring $R[T]$ is also hopfian. In this paper we generalize Varadarajan's result by proving the following:

Theorem. *Let R be a ring. Suppose there is an integer $n > 1$ such that $a^n = a$ for every a in R. Then R hopfian implies the ring $R[T]$ is hopfian.*

Note that any such ring is commutative by a well-known result of Jacobson ([Theorem 3.1.2 in [3]). Also, 0 is the only nilpotent element in R.

If B is a Boolean ring then $b^2 = b \,\forall\, b \in B$. Hence for such rings our theorem reduces to that of Varadarajan ([5], Theorem 1). By the hypothesis of our above theorem and in view of Jacobson's theorem, it follows that our ring R will be automatically commutative. We also remark that there are numerous rings R which satisfy the hypothesis of the above theorem but are neither Boolean nor Noetherian. For instance, let B be any Boolean ring and $GF(p)$ be a finite field of characteristic p. Then the product ring $R = B \times GF(p)$ has the property that zero element is the only nilpotent element in R. We also know that if the order of the field $GF(p)$ is $n = p^m$, then $a^n = a \,\forall\, a \in GF(p)$. On the other hand, trivially, $b^n = b \,\forall\, b \in B$. Therefore if $x = (a,b) \in B \times GF(p)$, then $x^n = (a^n, b^n) = (a,b) = x$.

As pointed out in [5], there are uncountably many Boolean rings B which are not Noetherian. We observe that $\forall\, p > 2$, the ring $GF(p)$ can not be a homomorphic image of B, and, 0 and 1 are the only idempotent elements of $GF(p)$. Therefore from ([5], Theorem 2), the product ring $R = B \times GF(p)$ is hopfian (for $p = 2$, one can directly see that R is hopfian). Thus the class of rings $R = B \times GF(p)$, $p > 2$, is hopfian but neither Boolean nor Noetherian.

We exploit the notations used in [5]. For a given ring R, let $R[T]$ denote the polynomial ring over the ring R. We put $I = \{p(T)\colon p(0) = 0\}$. Then, as a Z-module, $R[T]$ decomposes as $R[T] = \mathrm{R} \oplus I$, where R and I both are also R-modules. Any ring homomorphism $f\colon R[T] \to R[T]$, viewed as a Z-omomorphism, determines four Z-homomorphisms $\phi_{11}\colon R \to R$, $\phi_{12}\colon I \to R$, $\phi_{21}\colon R \to I$ and $\phi_{22}\colon I \to I$. Thus if an element of $R[T]$ is written as a column vector $\begin{pmatrix} r \\ p(T) \end{pmatrix}$, then

$$f\begin{pmatrix} r \\ p(T) \end{pmatrix} = \begin{pmatrix} \phi_{11}(r) + \phi_{12}(p(T)) \\ \phi_{21}(r) + \phi_{22}(p(T)) \end{pmatrix}, \tag{$*$}$$

and obviously the product in the ring $R[T]$ is given by the following formula:

$$\begin{pmatrix} r \\ p(T) \end{pmatrix} \cdot \begin{pmatrix} s \\ qT() \end{pmatrix} = \begin{pmatrix} r \cdot s \\ r \cdot q(T) + s \cdot p(T) + p(T) \cdot q(T) \end{pmatrix}. \tag{$**$}$$

Using the fact that f is a ring homomorphism, applying f on both sides of the above formula and substituting the values from ($*$) in terms of ϕ's, one can find how these additive homomorphisms behave with respect to the multiplication of the ring $R[T]$. For example taking $p(T) = 0 = q(T)$ in ($**$), we find that $\phi_{11}(r \cdot s) = \phi_{11}(r) \cdot \phi_{11}(s)$. In fact ϕ_{11} turns out to be the restriction of f to the ring R in view of the hypothesis on the ring R. Indeed the restriction f/R of f maps R into itself. We have lemma 1.

Lemma 1. *If for a given ring R, there is a positive integer $n > 1$, such that $a^n = a$, $\forall a \in R$, then any ring homomorphism $f\colon R[T] \to R[T]$ will map the ring R into itself, i.e., $\phi_{21} = 0$.*

Proof. If for some $a \in R$, $f(a)$ is a non-constant polynomial of degree, say $k > 0$, then $f(a) = b_0 + b_1 T + \cdots + b_k T^k$, where $b_k \neq 0$. Now $a^n = a$ in R implies $(f(a))^n = f(a)$. Since $k > 1$, the coefficient of T^{kn} in the LHS, i.e. $b_k^n = b_k = 0$, a contradiction.

The behaviour of $\phi_{22}\colon I \to I$ with respect to multiplication by elements of R depends on ϕ_{11}. We have, for every $h(T)$ in $R[T]$,

$$\phi_{22}(a \cdot h(T)) = \phi_{11}(a) \cdot \phi_{22} h(T).$$

This follows by applying f on both sides of the product

$$\begin{pmatrix} a \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ h(T) \end{pmatrix} = \begin{pmatrix} 0 \\ a \cdot h(T) \end{pmatrix}.$$

We note that I is an ideal of $R(T)$ and $\phi_{22}\colon I \to I$ is an additive homomorphism which need not respect the multiplication of I. In fact we have the following Lemma.

Lemma 2. *For every integer $j > 0$, $\phi_{22}(T^j) = \phi_{22}(T) \cdot h(T)$ where $h(T) \in R[T]$.*

Proof. It is trivial for $j = 1$. Hence to prove it by induction suppose $j > 1$ and $\phi_{22}(T^{j-1}) = \phi_{22}(T) \cdot h'(T)$ where $h'(T) \in R[T]$. Now consider the product $\begin{pmatrix} 0 \\ T^j \end{pmatrix} = \begin{pmatrix} 0 \\ T \end{pmatrix} \cdot \begin{pmatrix} 0 \\ T^{j-1} \end{pmatrix}$ and apply f on both sides in terms of ϕ's This gives us

$$\phi_{22}(T^j) = \phi_{12}(T)\phi_{22}(T^{j-1}) + \phi_{12}(T^{j-1}) \cdot \phi_{22}(T) + \phi_{22}(T) \cdot \phi_{22}(T^{j-1}).$$

Substituting the value of $\phi_{22}(T^{j-1})$ by inductive hypothesis yields the result.

PROPOSITION 3

Suppose R satisfies the hypothesis of the theorem and $f:R[T]\to R[T]$ is an onto ring homomorphism. Then $\phi_{22}(T)=u.T$, where u is a unit of R.

Proof. Suppose $\phi_{22}(T)=b_1T+b_2T^2+\cdots+b_kT^k=T\cdot h(T)$, say. Since the ring homomorphism $f:R[T]\to R[T]$ is onto, the Z-homomorphism $\phi_{22}:I\to I$ must be onto. Hence there is a polynomial $p(T)=a_1T+a_2T^2+\cdots+a_lT^l$ such that $\phi_{22}(p(T))=T$. Using the properties of ϕ_{22} with respect to the scalar multiplication and Lemma 2, we find that $\phi_{22}(p(T))=\phi_{22}(T)\cdot q(T)$ where $q(T)\in R(T)$. Therefore

$$(b_1T+b_2T^2+\cdots+b_kT^k)q(T)=T.$$

This means $T\cdot(h(T)\cdot q(T)-1)=0$, i.e., $h(T)=b_1+b_2T\ldots b_kT^{k-1}$ is invertible in $R[T]$. By the well-known fact from commutative algebra (see [1], p. 10, exer. 2), this implies that b_1 is a unit and $b_2,\ldots,b_k$ are all nilpotent elements of R. Since nil radical of R is trivial, $h(T)=b_1$, i.e., $\phi_{22}(T)=b_1T$ where b_1 is a unit.

From the above proposition and induction on j we have the following.

COROLLARY 4

For each $j>0$, $\phi_{22}(T^j)$ is a polynomial in T of degree j with leading coefficient a unit.

The following is an important step in the proof of the main theorem.

Lemma 5. Suppose R satisfies the hypotheses of the theorem and $f:R[T]\to R[T]$ is an onto ring homomorphism. Then the restriction $f|R:R\to R$ is onto.

Proof. It suffices to show that ϕ_{11} is onto. For this let $r\in R$. By Lemma 3 we know that there is a unit u in R such that $\phi_{22}(T)=u.T$. Since ϕ_{22} is onto, there is a polynomial $p(T)\in I$ such that $\phi_{22}(p(T))=r.u.T$. Suppose $p(T)=c_1T+c_2T^2+\cdots+c_kT^k$, $c_k\neq 0$. Then we get

$$\phi_{11}(c)\phi_{22}(T)+\cdots+\phi_{11}(c_k)\phi_{22}(T^k)=r.u.T.$$

By the preceding Corollary $\phi_{22}(T^k)$ is a polynomial of degree k and leading coefficient a unit. Hence if $k>1$, then $u.\phi_{11}(c_k)=0$ where u is a unit, which means $\phi_{11}(c_k)=0$. Then successively we get $\phi_{11}(c_{k-1})=0=\cdots=\phi_{11}(c_2)$, and $\phi_{11}(c).uT=ruT$, which implies that $\phi_{11}(c)=r$.

Proof of the Theorem. Let $f:R[T]\to R[T]$ be an onto ring homomorphism. Then by Lemma 4, the ring homomorphism $f|R:R\to R$ is onto. Since R is a hopfian ring, $f|R$ is an isomorphism. Next we prove that f is injective. Suppose $f\begin{pmatrix} r \\ p(T)\end{pmatrix}=0$. This means $\begin{pmatrix}\phi_{11}(r)+\phi_{12}(p(T)) \\ \phi_{22}(p(T))\end{pmatrix}=0$. It follows that $\phi_{22}(p(T))=0$. Now this last fact implies that $p(T)=0$ because otherwise the leading coefficient a_k of $p(T)$ is nonzero; then the leading coefficient of $\phi_{22}(p(T))$ must be $u\cdot a_k$ where u is a unit in R. Since $u\cdot a_k\neq 0$, we have a contradiction. Now $p(T)=0$ implies $\phi_{12}(p(T))=0$ which means $\phi_{11}(r)=0$. But then $f(r)=\phi_{11}(r)=0$ which means $r=0$, as $f|R\to R$ is an isomorphism. Thus $\begin{pmatrix} r \\ p(T)\end{pmatrix}=0$. Therefore f is a ring isomorphism, proving that $R[T]$ is hopfian.

Acknowledgement

The author is thankful to the UGC-CSIR for the award of JRF when this work was done. He is also thankful to K Varadarajan for his preprints of related papers.

References

[1] Atiyah M F and MacDonald I G, *Introduction to commutative algebra* (Addison Wesley) (1969)
[2] Deo S and Varadarajan K, Hopfian and cohopfian zero-dimensional spaces, *J. Ramanujan Math. Soc.* **9** (1994) 177–202
[3] Herstein I N, *Noncommutative rings, The carus mathematical monographs* (1973) monograph series No. 15
[4] Varadarajan K, Hopfian and cohopfian objects, *Publ. Mat.* **36** (1992) 293–317
[5] Varadarajan K, *On hopfian rings* (to appear in *Acta Math.*)

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 137–149.

Immersions in a symplectic manifold

MAHUYA DATTA

International Centre for Theoretical Physics, P.O. Box 586, 34100 Trieste, Italy
Department of Pure Mathematics, University College of Science, 35, P. Barua Sarani, Calcutta 700 019, India

MS received 2 August 1996; revised 10 February 1998

Abstract. In this paper we give a homotopy classification of symplectic isometric immersions following Gromov's h-principle theorem.

Keywords. Symplectic immersions; h-principle.

1. Introduction

Let (N, σ) be a smooth symplectic manifold and M a manifold with a closed C^∞ 2-form ω on it. A smooth immersion $f\colon (M, \omega) \to (N, \sigma)$ is called *symplectic isometric* (or simply *symplectic*) if f pulls back σ onto ω. The differential of f gives rise to a bundle monomorphism $F\colon TM \to TN$ such that $F^*\sigma = \omega$. Moreover, the underlying map $M \to N$ of F, which is f itself here, pulls back the de Rham cohomology class of σ onto that of ω. So a natural question would be if the existence of such a bundle map ensures the existence of a symplectic immersion. Gromov proves that when dim $M <$ dim N then the question has an affirmative answer. We prove here further that the space of symplectic immersions is the same up to homotopy as the space of those bundle maps. Let Symp (M, N) denote the space of symplectic immersions of M into N with C^∞ compact-open topology and Symp_0 (TM, TN) denote the associated space of bundle monomorphisms $F\colon TM \to TN$ with C^0 compact-open topology; i.e. F satisfies $F^*\sigma = \omega$ and its underlying (continuous) map f satisfies the cohomology condition $f^*[\sigma] = [\omega]$, where $[\sigma]$ and $[\omega]$ denote respectively the de Rham cohomology classes of σ and ω.

The main theorem may now be stated as follows.

Theorem 1.1. ([1], p. 334–335) *If dim $M <$ dim N then the differential map d: $\mathrm{Symp}(M, N) \to \mathrm{Symp}_0(TM, TN)$ is a weak homotopy equivalence. In fact, symplectic immersions satisfy the C^0-dense parametric h-principle in the space of continuous maps f: $M \to N$ which pull back the cohomology class of σ onto that of ω.*

It is interesting to note that when dim $N = 2$dim M, taking ω equal to zero we obtain the following result of Lees [2].

COROLLARY 1.2

The Lagrangian immersions satisfy C^0-dense parametric h-principle.

In the next theorem we state the relative version of the above h-principle.

Theorem 1.3. *Let $F: T(\mathrm{Op}A) \to TN$ be a bundle monomorphism such that $F^*\sigma = \omega$, where A is a compact set in M, and let the underlying map f be a symplectic immersion on a neighbourhood of a compact set $B \subset A$. If the relative cohomology class $[f^*\sigma - \omega]$ vanishes in $H^2(A, B)$ then F can be homotoped to a symplectic immersion such that the homotopy remains constant in a neighbourhood of B.*

It should be remarked that Gromov studied in [1, § 3.4.2] a more general problem, namely the h-principle of σ-regular isometric immersions for an arbitrary closed form σ. The general theorem arises from the h-principle of some auxiliary sheaf which comes as the solution sheaf of an infinitesimally invertible differential operator, and Gromov proves this by using sophisticated machinery like the Nash–Moser implicit function theorem along with his sheaf theoretical techniques. The maps into a symplectic manifold (N, σ) automatically satisfy the σ-regularity condition because of the non-degeneracy of the 2-form σ. Moreover, we may avoid the generalized implicit function theorem. Instead, we use Moser's Theorem on stability of symplectic forms, which says that if two symplectic forms on a compact manifold are homotopic within the same cohomology class then they are isotopic. Throughout this paper we shall extensively use different consequences of this stability theorem.

The proof of the theorems is based on sheaf theoretic techniques. The sheaf of symplectic isometric immersions arises as the solution space of a closed partial differential relation [1, p. 2]. To apply the above-mentioned technique one starts with a topological solution of the diffeferential relation and embeds the manifold (M, ω) in a symplectic manifold (M', ω') such that ω' restricts to ω on M, so that the restrictions of symplectic isometric immersions $(M', \omega') \to (N, \sigma)$ to M are isometric with respect to ω. We refer to the sheaf of symplectic immersions $M' \to N$ as the extension sheaf. An important requirement for the applicability of the sheaf theoretic techniques is the existence of a microflexible extension (see § 2). However, as we shall see in Example 3.3, this is not the case here. Noting that the symplectic immersion $(M', \omega') \to (N, \sigma)$ are in 1–1 correspondence with the Lagrangian sections of the product symplectic manifold $(M' \times N, \sigma - \omega')$ and that with the cohomology condition on the topological solution we can reduce the problem to the case where $\sigma - \omega'$ is exact, we pass on to the auxiliary sheaf of exact Lagrangian sections of $M' \times N \to M'$. This sheaf has the same homotopy type as the sheaf of Lagrangian section of $M' \times N$ and, moreover, it is microflexible. Another key point is to note that Hamiltonian diffeotopies on (M', ω') act on the auxiliary sheaf and sharply move M in M' [1, p. 82]. Thus we obtain h-principle for the auxiliary sheaf and hence the h-principle for symplectic immersions. Following the proof we also obtain

COROLLARY 1.4

If the symplectic form on N is exact, namely $\sigma = \mathrm{d}\tau$, and if $\dim N = 2\dim M$, *then the space of τ-exact Lagrangian immersions satisfy (everywhere C^0-dense) h-principle.*

For any undefined term we refer to [1].

2. Brief review of the sheaf theoretic results

We now briefly describe the sheaf theoretic techniques to prove parametric h-principle. Let Φ denote the sheaf of solutions of some rth order partial differential relation $\mathscr{R} \subset J^r(M, N)$ defined for C^r-maps $M \to N$, and Ψ the sheaf of sections of the r-jet bundle

$J^r(M,N)\to M$ with images in $\mathscr{R}$. The natural topologies on $\Phi(U)$ and $\Psi(U)$ are respectively the C^r and C^0 compact open topologies.

DEFINITION 2.1

The solution sheaf Φ and the relation $\mathscr{R}$ are said to satisfy parametric h-principle if the r-jet map $j^r:\Phi\to\Psi$ is a weak homotopy equivalence.

Before proceeding further we state some general definitions and results on topological sheaves.

DEFINITION 2.2

let $\mathscr{F}$ be a topological sheaf over M and A be a compact set in M. The symbol $\mathscr{F}(A)$ will denote the space of maps which are defined over some neighbourhood of A in M; in fact it is the direct limit of the spaces $\mathscr{F}(U)$ where U runs over all the open sets containing A. A map $f:P\to\mathscr{F}(A)$ on a polyhedron P is called continuous if there exists an open set $U\supset A$ such that each f_p is defined over U and the resulting map $P\to\mathscr{F}(U)$ is continuous with respect to the given topology on $\mathscr{F}(U)$.

DEFINITION 2.3

A topological sheaf $\mathscr{F}$ over M is flexible if the restriction maps $\mathscr{F}(A)\to\mathscr{F}(B)$ are Serre fibrations for every pair of compact sets (A,B), $A\supset B$. The restriction map $\mathscr{F}(A)\to\mathscr{F}(B)$ is called a microfibration if given a continuous map $f'_0: P\to\mathscr{F}(A)$ on a polyhedron P and a homotopy f_t, $0\leqslant t\leqslant 1$, of $f'_0|_B$ there exists an $\varepsilon>0$ and a homotopy f'_t of f'_0 such that $f'_t|\mathrm{Op}B=f_t$ for $0\leqslant t\leqslant\varepsilon$. If for any pair of compact sets the restriction morphism is a microfibration then the sheaf $\mathscr{F}$ is called microflexible.

DEFINITION 2.4

A sheaf homomorphism $f:\mathscr{F}\to\mathscr{G}$ is a local weak homotopy equivalence if for each $x\in M$ the homomorphism $f(x):\mathscr{F}(x)\to\mathscr{G}(x)$ is a weak homotopy equivalence.

Theorem 2.5. (sheaf homomorphism theorem [1, p. 77]) *Let $\mathscr{F}$ and $\mathscr{G}$ be two topological sheaves defined on a manifold M and let $f:\mathscr{F}\to\mathscr{G}$ be a morphism. If both sheaves are flexible and if f is a local weak homotopy equivalence then f is a weak homotopy equivalence.*

So to prove parametric h-principle for a relation $\mathscr{R}$ it suffices to show that the sheaves Φ and Ψ (as defined above) are flexible and the r-jet map $j^r:\Phi\to\Psi$ is a local weak homotopy equivalence. For any partial differential relation $\mathscr{R}$ the sheaf Ψ is always flexible [1, p. 40]. But to prove flexibility of Φ we need to impose certain extensibility conditions on $\mathscr{R}$.

Let M be embedded in a higher dimensional manifold M' and let $\mathscr{R}'$ be a relation on M'. We denote the corresponding sheaf of solutions by Φ'.

DEFINITION 2.6

Φ' is said to be an *extension* of Φ if the inclusion of M in M' induces a restriction homomorphism $\alpha:\Phi'|_M\to\Phi$; moreover, $\alpha(x)$ is a surjection for each $x\in M$.

This means that if we restrict a solution of $\mathscr{R}'$ to M we obtain a solution of $\mathscr{R}$ and moreover every local solution of $\mathscr{R}$ can be lifted to a local solution of $\mathscr{R}'$.

Now, for a pair of compact subsets (A,B) in M we define the space $\Gamma(A,B)$ of compatible pairs of solutions inside $\Phi'(B)\,\Phi(A)$. This set consists of all pairs (f',f) such that $\alpha(f')=f|\mathrm{Op}B$.

DEFINITION 2.7

The extension Φ' will be called a microextension if the obvious map $\gamma:\Phi'(A)\to\Gamma(A,B)$ is a microfibration.

Now we explain the concept of diffeotopy sharply moving a submanifold in M'. It is worth recalling that the idea contained in this definition is a key point in the Smale–Hirsch immersion Theorem.

DEFINITION 2.8

We fix a metric d on M. An open set in M will be called 'small' if it is contained in a ball of small radius. A class of diffeotopies $\mathscr{D}$ on M' is said to sharply move M in M' if given any hypersurface S lying in a small open set of M and given any positive number ε we can obtain a diffeotopy $\{\delta_t\}$ in $\mathscr{D}$ which satisfies the following conditions:

(a) δ_0 is the identity map,
(b) each δ_t is identity outside an ε-neighbourhood of S in M,
(c) $d(\delta_1(S), M) > r$ for some $r > 0$.

Gromov gives the following sufficient condition for flexibility of Φ in his main Lemma [1, p. 82] and microextension Theorem [1,. 85].

Theorem 2.9. *If Φ admits a microextension Φ' which is microflexible and if there exists a class of acting diffeotopy on Φ' which sharply moves M in M' then Φ is a flexible sheaf.*

3. Construction of an extension sheaf

Let (M,ω) and (N,σ) be as in § 1. Then the symplectic immersions $(M,\omega)\to(N,\sigma)$ correspond to the partial differential relation $\mathscr{R}\subset J^{(1)}(M,N)$ consisting of 1-jets $j^1_x f$, $x\in M$, of germs of local immersions f such that $f^*\sigma=\omega$ at x. Let Ψ denote the sheaf of bundle monomorphisms $F:TM\to TN$ which pull back the form σ onto ω. This may be identified with the sheaf of sections of the relation $\mathscr{R}$. To obtain an extension of $\mathscr{R}$, we first embed (M,ω) isometrically into a symplectic manifold (M',ω'). We start with an $F:TM\to TN$ in $\Psi(M)$ whose underlying map is $f:M\to N$ and consider the bundle f^*TN/TM over M. Without loss of generality we may assume that F is smooth. Observe that the total space of the bundle, which we denote by X, has the same dimension as N. We extend F to a bundle morphism $F':TX|M\to TN$ such that F' maps fibres of $TX|M$ isomorphically onto the fibres of TN. Since the form $F'^*\sigma$ restricts to the closed form ω on M, it extends to a closed form ω' on some neighbourhood M' of M in X. M' may be taken to be a tubular neighbourhood of M in X so that the inclusion i: $M\to M'$ is a homotopy equivalence. Since $F'^*\sigma$ is non-degenerate so is ω'. Thus, (M,ω) is isometrically embedded in the symplectic manifold (M',ω').

We shall denote the sheaf of symplectic isometric immersions of (M,ω) in (N,σ) by $\mathscr{S}$ and that of (M',ω') in (N,σ) by $\mathscr{S}'$. Let $\mathscr{R}'$ denote the space of 1-jets of germs of symplectic immersions of (M',ω') in (N,σ) and Ψ' the sheaf of section of $\mathscr{R}'$.

PROPOSITION 3.1

$\mathscr{S}'$ is an extension of $\mathscr{S}$.

Proof. It is easy to see that the isometric embedding of (M,ω) in (M',ω') induces a morphism $\alpha:\mathscr{S}'|_M \to \mathscr{S}$. To prove that $\alpha(x):\mathscr{S}'(x)\to\mathscr{S}(x)$ is onto we start with a local symplectic immersion f at a point $x\in M$. Let $\bar{f}$ be any extension of f to a local immersion in M'. Then, since the dimension of M' is the same as the dimension of N, the form $\bar{\omega}=\bar{f}^*\sigma$ is a symplectic form. Now the two linear symplectic forms $\bar{\omega}_x$ and ω'_x defined on T_xM' coincide on the subspace T_xM. Hence there exists a linear isomorphism l of T_xM' which pulls back $\bar{\omega}_x$ onto ω'_x and keeps T_xM pointwise fixed. We consider the germ of a local map f' whose 1-jet at x equals to $j^1_x\bar{f}\circ l$ so that $j^1_xf'\in\mathscr{R}'$. By construction the jet j^1_xf' projects onto $j^1_xf\in\mathscr{R}$. Moreover we may assume without loss of generality that f' extends f. So we have the following:

- $f'^*\sigma=\omega'$ at x.
- f' equals f on $U\cap M$, where U is the domain of f. Hence, pullbacks of both the forms $f'^*\sigma$ and ω' are the same.

Therefore, by the relative Poincaré Lemma, we obtain a 1-form φ on a neighbourhood, say $\tilde{U}$, of x in U such that $d\varphi=f'^*\sigma-\omega'$ and $\varphi|\tilde{U}\cap M=0$. Now, by applying Moser's theorem [3] we get a diffeomorphism δ on a neighbourhood, say U', of x in $\tilde{U}$, such that $\delta^*(f'^*\sigma)=\omega'$, $\delta|U'\cap M$ is identity, and $d\delta_x=id$. Then $f'\circ\delta$ is the required extension of f. ■

PROPOSITION 3.2

The 1-jet map $j^1:\mathscr{S}\to\Psi$ is a local weak homotopy equivalence.

Proof. The main point is to observe that an infinitesimal solution can be deformed to a local solution of the relation $\mathscr{R}$. To see this, we start with an infinitesimal solution f at x, so that $f^*\sigma=\omega$ at x. Proceeding as in the proof of the above lemma we may extend f to a map f' on a neighbourhood of x in M' such that $f'^*\sigma=\omega'$ at x. Set $f'^*\sigma=\omega''$. Since $\omega''=\omega'$ at x, therefore ω'' is symplectic on a neighbourhood of x. Applying Moser's theorem we get a local isotopy δ_t at x such that $\delta_1^*\omega''=\omega'$. Moreover, the homotopy keeps x and T_xM' pointwise fixed. Defining $\bar{f}=f'\circ\delta|_M$ on $\mathrm{Op}\,x$ we observe that $\bar{f}$ is a symplectic immersion and it is homotopic to f in the space of infinitesimal solutions of $\mathscr{R}$.

The remaining part of the proof is now a routine work in view of the above observation (and hence we omit it here). □

However, it can be seen from the following example that the extension sheaf $\mathscr{S}'$ is not microflexible.

Example 3.3. Consider the standard embedding of the closed unit disc in $\mathbb{R}^2$. If we deform it near the boundary by pushing it inside then it (the homotopy) cannot be extended symplectically on the whole of the disc.

This phenomenon may be explained as follows: If f_0 is a symplectic immersion over $\mathrm{Op}\,A$ and f_t a homotopy of f_0 such that $f_t|\mathrm{Op}B$ is a symplectic immersion, then the relative cohomology class of $f_t^*\sigma-\omega$ in $H^2(A,B)$ determines the obstruction to

extending $f_t|\mathrm{Op}B$ to Op A as symplectic immersions. If the cohomology class $[f_t^*\sigma - \omega] = 0 \in H^2(A, B)$ then there exists a smooth family of 1-forms α_t which vanishes on $\mathrm{Op}B$ and $f_t^*\sigma - \omega = d\alpha_t$. Then Moser's stability Theorem applies and we can lift $f_t|\mathrm{Op}B$ over A as symplectic immersion.

Since $\mathscr{S}'$ is not microflexible we cannot apply the sheaf theoretic techniques (described in § 2) on it. However, we shall see in the next section that there exists a topological sheaf on M' naturally associated to a subspace of the space of symplectic immersions which do satisfy microflexibility and has the same homotopy type as $\mathscr{S}'$.

4. Sheaf of exact Lagrangian sections

Throughout this section we assume that both σ and ω' are exact symplectic forms. Let p_1 and p_2 respectively denote the projections of $M' \times N$ onto the first and the second factor. The product form $p_2^*\sigma - p_1^*\omega'$ on $M' \times N$ is then an exact symplectic form. We denote it by $\sigma - \omega'$. Let τ be a 1-form such that $\sigma - \omega' = \mathrm{d}\tau$.

If $f: M' \to N$ is a symplectic isometric immersion then its graph map $g = (1, f)$: $M' \to M' \times N$ is a Lagrangian section of $(M' \times N,\ \sigma - \omega')$. Since $\sigma - \omega' = \mathrm{d}\tau$, the Lagrangian condition becomes equivalent to closeness of the form $g^*\tau$. It is easy to observe that the correspondence $f \mapsto g$ is bijective. We now construct the sheaf of exact Lagrangian sections as follows: This consists of pairs (g, φ), where $g: M' \to N$ is a section of the product bundle such that map $f = p_2 \circ g: M' \to N$ is an immersion, and φ is a function on M' satisfying $g^*\tau = \mathrm{d}\varphi$. (Such a g is called a τ-exact Lagrangian immersion.) We denote the sheaf of such pairs by $\mathscr{E}'$ and call it the sheaf of τ-exact Lagrangian sections. Observe that $\mathscr{S}'$ and $\mathscr{E}'$ are locally homotopically equivalent since the germ of a Lagrangian section at a point denotes a germ of an exact Lagrangian section; moreover the space of primitives φ for a τ-exact Lagrangian section g (meaning that φ satisfies the relation $g^*\tau = \mathrm{d}\varphi$) is isomorphic to $\mathbb{R}$. Consequently, the sheaf of sections corresponding to the relation, of which $\mathscr{E}'$ is the solution sheaf, has the same homotopy type as Ψ'. We now prove

PROPOSITION 4.1

The sheaf $\mathscr{E}'$ of τ-eact Lagrangian sections is microflexible.

Proof. Let (A, B) be a pair of compact sets in M'. Let g' be a τ-exact Lagrangian section over a A (meaning that it is defined on a neighbourhood of A) such that $g'^*\tau = \mathrm{d}\varphi'$ for a 0-form φ', and (g_t, φ_t) a homotopy of $(g', \varphi')|\mathrm{Op}B$ in $\mathscr{E}'$.

We first prove the following simple lemma.

Lemma 4.2 *Let g_t be a homotopy of τ-exact Lagrangian sections. If g_0 is also a τ'-exact Lagrangian section for some 1-form τ' on $M' \times N$ satisfying $\sigma - \omega' = \mathrm{d}\tau'$, then every g_t is a τ'-exact Lagrangian section.*

Proof. Two such forms τ and τ' differ by a closed 1-form c on M'. So, we have the following relation

$$g_t^*\tau' = g_t^*\tau + g_t^*c$$

for every t. Then, by hypothesis, g_0^*c is an exact form. Since c is closed, g_t^*c is also exact. Consequently $g_t^*\tau'$ is exact for each t. ■

$$= \frac{(z_0 + z - z_0)^{\Gamma_\Psi + 1}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi + 1)}\beta^{\Gamma_\Psi + 1}a^{n_l(\Gamma_\Psi + 1)}}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1)}}$$

$$+ O\left(\frac{1}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1) - 3}}\right)$$

$$= \frac{z_0{}^{\Gamma_\Psi + 1}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi + 1)}\beta^{\Gamma_\Psi + 1}a^{n_l(\Gamma_\Psi + 1)}}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1)}}$$

$$+ \frac{(\Gamma_\Psi + 1)z_0{}^{\Gamma_\Psi}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi + 1)}\beta^{\Gamma_\Psi + 1}a^{n_l(\Gamma_\Psi + 1)}}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1) - 1}}$$

$$+ \frac{(\Gamma_\Psi + 1)c_2 z_0{}^{\Gamma_\Psi - 1}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi + 1)}\beta^{\Gamma_\Psi + 1}a^{n_l(\Gamma_\Psi + 1)}}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1) - 2}}$$

$$+ O\left(\frac{1}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1) - 3}}\right)$$

$$= \frac{z_0{}^{\Gamma_\Psi - 1}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi + 1)}\beta^{\Gamma_\Psi + 1}a^{n_l(\Gamma_\Psi + 1)}}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1)}}$$

$$+ \frac{(\Gamma_\Psi + 1)z_0{}^{\Gamma_\Psi}(-1)^{(\Gamma_\Psi - n_l + 1)(\Gamma_\Psi + 1)}\beta^{\Gamma_\Psi + 1}a^{n_l(\Gamma_\Psi + 1)}}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1) - 1}} + O\left(\frac{1}{(z - z_0)^{\Gamma_\Psi(\Gamma_\Psi + 1) - 2}}\right).$$

Hence

$$g(z) = \frac{N}{D} = \frac{(\Gamma_\Psi)^{\Gamma_\Psi}}{z_0 a^{n_l}(-1)^{\Gamma_\Psi - n_l + 1}}(1 + 0(z - z_0)^2).$$

Therefore, $g(z_0) \neq 0$, $\neq \infty$ in a neighbourhood of z_0 and $g'(z_0) = 0$. Thus,

$$\bar{N}_{1)}(r, f) \leqslant N_0\left(r, \frac{1}{g'}\right) + \log^+ r, \tag{13}$$

where $N_0(r, (1/g'))$ denotes the counting function of zeros of $g'(z)$ which are not zeros $g(z)$. Since

$$g(z) = \frac{(z\Psi' - \Psi)^{\Gamma}\Psi}{z^{\Gamma}\Psi^{+1}(z - \Psi)^{\Gamma}\Psi^{+1}}$$

$$\log g(z) = \Gamma_\Psi \log(z\Psi' - \Psi) - (\Gamma_\Psi + 1)\log z - (\Gamma_\Psi + 1)\log(z - \Psi),$$

$$\frac{g'(z)}{g(z)} = \frac{\Gamma_\Psi z\Psi''}{z\Psi' - \Psi} - \frac{\Gamma_\Psi + 1}{z} - \frac{(\Gamma_\Psi + 1)(\Psi' - 1)}{\Psi - z},$$

$$\frac{g'(z)}{g(z)} = \frac{\Gamma_\Psi z^2\Psi''(\Psi - z) - (\Gamma_\Psi + 1)(z\Psi' - \Psi)(\Psi - z) - (\Gamma_\Psi + 1)(\Psi' - 1)(z\Psi' - \Psi)z}{z(\Psi - z)(z\Psi' - \Psi)}$$

$$\frac{g(z)}{g'(z)} = \frac{z(\Psi - z)(z\Psi' - \Psi)}{\Gamma_\Psi z^2\Psi''(\Psi - z) - (\Gamma_\Psi + 1)(z\Psi' - \Psi)(\Psi - z) - (\Gamma_\Psi + 1)(\Psi' - 1)(z\Psi' - \Psi)z}$$

$$= \frac{-1}{\Gamma_\Psi + 1}z + O(z^2)$$

is the Taylor series of $g(z)/g'(z)$ in the neighbourhood of the origin.

If C_λ is the first nonzero term of Taylor series of $g(z)/g'(z)$ in the neighbourhood of origin, then

$$\log|C_\lambda| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{g(re^{i\phi})}{g'(re^{i\phi})}\right| d\phi + N\left(r,\frac{g}{g'}\right) - N\left(r,\frac{g'}{g}\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{g(re^{i\phi})}{g'(re^{i\phi})}\right| d\phi - \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{g'(re^{i\phi})}{g(re^{i\phi})}\right| d\phi$$

$$+ N(r,g) - N(r,g') + N\left(r,\frac{1}{g'}\right) - N\left(r,\frac{1}{g}\right)$$

$$= m\left(r,\frac{g}{g'}\right) - m\left(r,\frac{g'}{g}\right) - \bar{N}(r,g) + N_0\left(r,\frac{1}{g'}\right) - \bar{N}\left(r,\frac{1}{g}\right).$$

Therefore

$$N_0\left(r,\frac{1}{g'}\right) \leqslant \bar{N}(r,g) + \bar{N}\left(r,\frac{1}{g}\right) + m\left(r,\frac{g'}{g}\right) + \log|C_\lambda|. \tag{14}$$

From the expression of $g(z)$ it is clear that any zero or pole of $g(z)$ can only occur at zeros of $\Psi(z) - z$, $z = 0$ and poles of $f(z)$ of order greater than one and zeros of $z\Psi' - \Psi$. Therefore

$$\bar{N}(r,g) + \bar{N}\left(r,\frac{1}{g}\right) < \bar{N}_{(2}(r,f) + \bar{N}\left(r,\frac{1}{\Psi - z}\right) + \bar{N}\left(r,\frac{1}{z\Psi' - \Psi}\right) + \log^+ r, \tag{15}$$

where $\bar{N}_{(2}(r,f)$ denotes the counting function of multiple poles of $f(z)$, each of them counted only once.

Since

$$\bar{N}_{(2}(r,f) = \bar{N}(r,f) - \bar{N}_{1)}(r,f),$$

We get

$$2\bar{N}_{1)}(r,f) \leqslant \bar{N}(r,f) + \bar{N}\left(r,\frac{1}{\Psi - z}\right) + \bar{N}\left(r,\frac{1}{z\Psi - \Psi}\right)$$

$$+ m\left(r,\frac{g'}{g}\right) + \log|C_\lambda| + 2\log^+ r$$

using (13), (14) and (15).

Lemma 6. *Let $f(z)$ and $\Psi(z)$ be as in Lemma* 4. *Then*

$$T(r,f) < \frac{3\alpha}{3\alpha - 2} N\left(r,\frac{1}{f}\right) + \frac{4}{3\alpha - 2} N\left(r,\frac{1}{\Psi - z}\right)$$

$$- \frac{2\alpha}{3\alpha - 2} N\left(r,\frac{1}{z\Psi - \Psi}\right) + S(r,f), \tag{16}$$

where $\alpha = 3n_1 - 2n$.

Proof. We have, be using Lemmas 4 and 5,

$$2\bar{N}(r,f) \leqslant N(r,f) + \bar{N}_{(1)}(r,f)$$

$$< T(r,f) + \bar{N}_{1)}(r,f)$$

$$< \frac{1}{\alpha}\bar{N}(r,f) + N\left(r,\frac{1}{f}\right) + \frac{1}{\alpha}N\left(r,\frac{1}{\psi-z}\right) - \frac{1}{\alpha}N\left(r,\frac{1}{z\Psi-\Psi}\right)$$

$$+ \bar{N}_{(1)}(r,f) + S(r,f)$$

$$< \frac{1}{\alpha}\bar{N}(r,f) + N\left(r,\frac{1}{f}\right) + \frac{1}{\alpha}N\left(r,\frac{1}{\psi-z}\right) - \frac{1}{\alpha}N\left(r,\frac{1}{z\Psi-\Psi}\right) + \frac{1}{2}\bar{N}(r,f)$$

$$+ \frac{1}{2}\bar{N}\left(r,\frac{1}{\psi-z}\right) + \frac{1}{2}N\left(r,\frac{1}{z\Psi'-\Psi}\right) + \frac{1}{2}m\left(r,\frac{g'}{g}\right) + \frac{1}{2}\log|C_\lambda|$$

$$+ \log^+ r + S(r,f),$$

$$\left(2 - \frac{2+\alpha}{2\alpha}\right)\bar{N}(r,f) < N\left(r,\frac{1}{f}\right) + \frac{2+\alpha}{2\alpha}N\left(r,\frac{1}{\Psi-z}\right)$$

$$- \left(\frac{2-\alpha}{2\alpha}\right)N\left(r,\frac{1}{z\Psi'-\Psi}\right) + \frac{1}{2}m\left(r,\frac{g'}{g}\right) + S(r,f),$$

or

$$\frac{3\alpha-2}{2\alpha}\bar{N}(r,f) < N\left(r,\frac{1}{f}\right) + \frac{2+\alpha}{2\alpha}N\left(r,\frac{1}{\Psi-z}\right) - \left(\frac{2-\alpha}{2\alpha}\right)N\left(r,\frac{1}{z\Psi'-\Psi}\right)$$
$$+ \frac{1}{2}m\left(r,\frac{g'}{g}\right) + S(r,f).$$

Hence

$$\bar{N}(r,f) < \frac{2\alpha}{3\alpha-2}N\left(r,\frac{1}{f}\right) + \frac{2+\alpha}{3\alpha-2}N\left(r,\frac{1}{\Psi+z}\right)$$
$$- \frac{2-\alpha}{3\alpha-2}N\left(r,\frac{1}{z\Psi'-\Psi}\right) + S(r,f). \tag{17}$$

Since

$$\frac{g'}{g} = \frac{\Gamma_\Psi z\Psi''}{z\Psi'-\Psi} - \frac{\Gamma_\Psi+1}{z} - \frac{(\Gamma_\Psi+1)(\Psi'-1)}{\Psi-z}$$

and, by Lemma 2,

$$m\left(r,\frac{g'}{g}\right) \leqslant m\left(r,\frac{(z\Psi'-\Psi)'}{z\Psi'-\Psi}\right) + m\left(r,\frac{1}{z}\right) + m\left(r,\frac{(\Psi-z)'}{\Psi-z}\right)$$
$$+ \log 3 + 2\log(\Gamma_\Psi+1) + \log\Gamma_\Psi = S(r,f).$$

Substituting (17) in (5), we get

$$\alpha T(r,f) < \frac{2\alpha}{3\alpha-2}N\left(r,\frac{1}{f}\right) + \frac{2+\alpha}{3\alpha-2}N\left(r,\frac{1}{\Psi-z}\right) - \frac{2-\alpha}{3\alpha-2}N\left(r,\frac{1}{z\Psi'-\Psi}\right)$$
$$+ \alpha N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{\Psi-z}\right) - N\left(r,\frac{1}{z\Psi'-\Psi}\right) + S(r,f)$$

$$< \frac{3\alpha^2}{3\alpha-2} N\left(r, \frac{1}{f}\right) + \frac{4\alpha}{3\alpha-2} N\left(r, \frac{1}{\Psi - z}\right) - \frac{2\alpha}{3\alpha-2} N\left(r, \frac{1}{z\Psi' - \Psi}\right) + S(r, f),$$

$$T(r, f) < \frac{3\alpha}{3\alpha-2} N\left(r, \frac{1}{f}\right) + \frac{4}{3\alpha-2} N\left(r, \frac{1}{\Psi - z}\right)$$

$$- \frac{2}{3\alpha-2} N\left(r, \frac{1}{z\Psi' - \Psi}\right) + S(r, f). \tag{18}$$

Proof of Theorem 1. From (18), we have

$$T(r, f) < \frac{3\alpha}{3\alpha-2} N\left(r, \frac{1}{f}\right) + \frac{4}{3\alpha-2} N\left(r, \frac{1}{\Psi - z}\right)$$

$$- \frac{2}{3\alpha-2} N\left(r, \frac{1}{z\Psi' - \Psi}\right) + S(r, f).$$

Changing from f to $f - z$, we have

$$\Psi_{f-z}(z) = M_l[f - z] + \sum_{j=1}^{l-1} a_j(z) M_j[f - z]$$

$$= \Psi_f(z) = \Psi(z)$$

and noting that

$$T(r, f) = T(r, f - z + z) \leqslant T(r, f - z) + T(r, z) \leqslant T(r, f - z) + O(\log r),$$

we get

$$T(r, f) < \frac{3\alpha}{3\alpha-2} N\left(r, \frac{1}{f - z}\right) + \frac{4}{3\alpha-2} N\left(r, \frac{1}{\Psi - z}\right) + S(r, f).$$

If the equations $f(z) = z$ and $\Psi(z) = z$ have only finite number of roots, we deduce

$$(1 + O(1))T(r, f) = O(\log r) \text{ as } r \to \omega$$

so that $f(z)$ is rational, which contradicts our hypothesis. This proves the theorem.

Remark. If $\Psi(z)$ is a homogeneous differential polynomial in $f^{(2)}$ of degree n, then $n = \alpha = n_1$.

Acknowledgement

The authors wish to thank the referee for his comments and careful observations. Lemma 3 was proved in the present improved form following the guidelines of the referee.

References

[1] Chuang Chi-tai, *On differential polynomials, Analysis of one complex* variables (Singapore 1987).

[2] Doeringer W, Exceptional values of differential polynomials. *Pacific J. Math.* **98** (1982) 55–62

[3] Gopalakrishna H S and Subhas S Bhoosnurmath, On the deficiencies of differential polynomials, *The Karnatak Univ. J. Science* **XVIII** (1973) 329–335

[4] Gross Fred, On fix-points of meromorphic functions, *Math. Scand.* **36** (1968) 230–231
[5] Hayman W K, *Meromorphic functions* (Oxford University Press) (1964)
[6] Sons L R, Deficiencies of monomials, *Math Z.* **III** (1969) 53–68
[7] Yang Lo, Normal families and fix-points of meromorphic functions, *Indiana Univ. Math. J.* **35** (1986) 171–191

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 133–136.

On the hopficity of the polynomial rings

S P TRIPATHI

Department of Mathematics, K. M. College, University of Delhi, Delhi 110007, India

MS received 19 January 1998

Abstract. In this paper we prove that if a ring R satisfies the condition that for some integer $n > 1, a^n = a$ for every a in R, then R a hopfian ring implies that the ring $R[T]$ of polynomials is also hopfian. This generalizes a recent result of Varadarajan which states that if R is a Boolean hopfian ring then the ring $R[T]$ is also hopfian. We show furthermore that there are numerous rings R satisfying the hypothesis of our theorem which are neither Boolean nor Noetherian.

Keywords. Hopfian rings.

Let R be a ring with $1 \neq 0$ and suppose $R[T]$ is the polynomial ring over R in one indeterminate T. Recall that a ring R is said to be hopfian (resp. cohopfian) if every onto (resp. one-one) ring homomorphism $f: R \to R$ is automatically an isomorphism (see [2] and [4] for details). The question whether or not R hopfian implies $R[T]$ hopfian is still unresolved. A Noetherian ring is easily seen to be hopfian (see [1], p. 78, exer. 1). By the Hilbert basis theorem R Noetherian implies the ring $R[T]$ is Noetherian and hence is hopfian. Therefore the general question about the hopficity of the ring $R[T]$ reduces to those rings R which are hopfian but not Noetherian. Using an interesting existence result proved in [2], Varadarajan has already pointed out that there exist such rings in abundance [5]. He has also proved the following result: If R is a Boolean hopfian ring then the ring $R[T]$ is also hopfian. In this paper we generalize Varadarajan's result by proving the following:

Theorem. *Let R be a ring. Suppose there is an integer $n > 1$ such that $a^n = a$ for every a in R. Then R hopfian implies the ring $R[T]$ is hopfian.*

Note that any such ring is commutative by a well-known result of Jacobson ([Theorem 3.1.2 in [3]). Also, 0 is the only nilpotent element in R.

If B is a Boolean ring then $b^2 = b \forall b \in B$. Hence for such rings our theorem reduces to that of Varadarajan ([5], Theorem 1). By the hypothesis of our above theorem and in view of Jacobson's theorem, it follows that our ring R will be automatically commutative. We also remark that there are numerous rings R which satisfy the hypothesis of the above theorem but are neither Boolean nor Noetherian. For instance, let B be any Boolean ring and $GF(p)$ be a finite field of characteristic p. Then the product ring $R = B \times GF(p)$ has the property that zero element is the only nilpotent element in R. We also know that if the order of the field $GF(p)$ is $n = p^m$, then $a^n = a \forall a \in GF(p)$. On the other hand, trivially, $b^n = b \forall b \in B$. Therefore if $x = (a, b) \in B \times GF(p)$, then $x^n = (a^n, b^n) = (a, b) = x$.

As pointed out in [5], there are uncountably many Boolean rings B which are not Noetherian. We observe that $\forall p > 2$, the ring $GF(p)$ can not be a homomorphic image of B, and, 0 and 1 are the only idempotent elements of $GF(p)$. Therefore from ([5], Theorem 2), the product ring $R = B \times GF(p)$ is hopfian (for $p = 2$, one can directly see that R is hopfian). Thus the class of rings $R = B \times GF(p)$, $p > 2$, is hopfian but neither Boolean nor Noetherian.

We exploit the notations used in [5]. For a given ring R, let $R[T]$ denote the polynomial ring over the ring R. We put $I=\{p(T): p(0)=0\}$. Then, as a Z-module, $R[T]$ decomposes as $R[T]=\mathrm{R}\oplus I$, where R and I both are also R-modules. Any ring homomorphism $f: R[T]\rightarrow R[T]$, viewed as a Z-omomorphism, determines four Z-homomorphisms $\phi_{11}: R\rightarrow R$, $\phi_{12}: I\rightarrow R$, $\phi_{21}: R\rightarrow I$ and $\phi_{22}: I\rightarrow I$. Thus if an element of $R[T]$ is written as a column vector $\begin{pmatrix} r \\ p(T)\end{pmatrix}$, then

$$f\begin{pmatrix} r \\ p(T)\end{pmatrix}=\begin{pmatrix}\phi_{11}(r)+\phi_{12}(p(T))\\ \phi_{21}(r)+\phi_{22}(p(T))\end{pmatrix}, \tag{$*$}$$

and obviously the product in the ring $R[T]$ is given by the following formula:

$$\begin{pmatrix} r \\ p(T)\end{pmatrix}\cdot\begin{pmatrix} s \\ qT()\end{pmatrix}=\begin{pmatrix} r\cdot s \\ r\cdot q(T)+s\cdot p(T)+p(T)\cdot q(T)\end{pmatrix}. \tag{$**$}$$

Using the fact that f is a ring homomorphism, applying f on both sides of the above formula and substituting the values from $(*)$ in terms of ϕ's, one can find how these additive homomorphisms behave with respect to the multiplication of the ring $R[T]$. For example taking $p(T)=0=q(T)$ in $(**)$, we find that $\phi_{11}(r\cdot s)=\phi_{11}(r)\cdot\phi_{11}(s)$. In fact ϕ_{11} turns out to be the restriction of f to the ring R in view of the hypothesis on the ring R. Indeed the restriction f/R of f maps R into itself. We have lemma 1.

Lemma 1. *If for a given ring R, there is a positive integer $n>1$, such that $a^n=a, \forall a\in R$, then any ring homomorphism $f: R[T]\rightarrow R[T]$ will map the ring R into itself, i.e., $\phi_{21}=0$.*

Proof. If for some $a\in R$, $f(a)$ is a non-constant polynomial of degree, say $k>0$, then $f(a)=b_0+b_1T+\cdots+b_kT^k$, where $b_k\neq 0$. Now $a^n=a$ in R implies $(f(a))^n=f(a)$. Since $k>1$, the coefficient of T^{kn} in the LHS, i.e. $b_k^n=b_k=0$, a contradiction.

The behaviour of $\phi_{22}: I\rightarrow I$ with respect to multiplication by elements of R depends on ϕ_{11}. We have, for every $h(T)$ in $R[T]$,

$$\phi_{22}(a\cdot h(T))=\phi_{11}(a)\cdot\phi_{22}h(T).$$

This follows by applying f on both sides of the product

$$\begin{pmatrix} a \\ 0\end{pmatrix}\cdot\begin{pmatrix} 0 \\ h(T)\end{pmatrix}=\begin{pmatrix} 0 \\ a\cdot h(T)\end{pmatrix}.$$

We note that I is an ideal of $R(T)$ and $\phi_{22}: I\rightarrow I$ is an additive homomorphism which need not respect the multiplication of I. In fact we have the following Lemma.

Lemma 2. *For every integer $j>0$, $\phi_{22}(T^j)=\phi_{22}(T)\cdot h(T)$ where $h(T)\in R[T]$.*

Proof. It is trivial for $j=1$. Hence to prove it by induction suppose $j>1$ and $\phi_{22}(T^{j-1})=\phi_{22}(T)\cdot h'(T)$ where $h'(T)\in R[T]$. Now consider the product $\begin{pmatrix} 0 \\ T^j\end{pmatrix}=\begin{pmatrix} 0 \\ T\end{pmatrix}\cdot\begin{pmatrix} 0 \\ T^{j-1}\end{pmatrix}$ and apply f on both sides in terms of ϕ's This gives us

$$\phi_{22}(T^j)=\phi_{12}(T)\phi_{22}(T^{j-1})+\phi_{12}(T^{j-1})\cdot\phi_{22}(T)+\phi_{22}(T)\cdot\phi_{22}(T^{j-1}).$$

Substituting the value of $\phi_{22}(T^{j-1})$ by inductive hypothesis yields the result.

PROPOSITION 3

Suppose R satisfies the hypothesis of the theorem and $f: R[T] \to R[T]$ is an onto ring homomorphism. Then $\phi_{22}(T) = u.T$, where u is a unit of R.

Proof. Suppose $\phi_{22}(T) = b_1 T + b_2 T^2 + \cdots + b_k T^k = T \cdot h(T)$, say. Since the ring homomorphism $f: R[T] \to R[T]$ is onto, the Z-homomorphism $\phi_{22}: I \to I$ must be onto. Hence there is a polynomial $p(T) = a_1 T + a_2 T^2 + \cdots + a_l T^l$ such that $\phi_{22}(p(T)) = T$. Using the properties of ϕ_{22} with respect to the scalar multiplication and Lemma 2, we find that $\phi_{22}(p(T)) = \phi_{22}(T) \cdot q(T)$ where $q(T) \in R(T)$. Therefore

$$(b_1 T + b_2 T^2 + \cdots + b_k T^k) q(T) = T.$$

This means $T \cdot (h(T) \cdot q(T) - 1) = 0$, i.e., $h(T) = b_1 + b_2 T \ldots b_k T^{k-1}$ is invertible in $R[T]$. By the well-known fact from commutative algebra (see [1], p. 10, exer. 2), this implies that b_1 is a unit and $b_2, \ldots, b_k$ are all nilpotent elements of R. Since nil radical of R is trivial, $h(T) = b_1$, i.e., $\phi_{22}(T) = b_1 T$ where b_1 is a unit.

From the above proposition and induction on j we have the following.

COROLLARY 4

For each $j > 0$, $\phi_{22}(T^j)$ is a polynomial in T of degree j with leading coefficient a unit.

The following is an important step in the proof of the main theorem.

Lemma 5. Suppose R satisfies the hypotheses of the theorem and $f: R[T] \to R[T]$ is an onto ring homomorphism. Then the restriction $f|R: R \to R$ is onto.

Proof. It suffices to show that ϕ_{11} is onto. For this let $r \in R$. By Lemma 3 we know that there is a unit u in R such that $\phi_{22}(T) = u.T$. Since ϕ_{22} is onto, there is a polynomial $p(T) \in I$ such that $\phi_{22}(p(T)) = r.u.T$. Suppose $p(T) = c_1 T + c_2 T^2 + \cdots + c_k T^k$, $c_k \neq 0$. Then we get

$$\phi_{11}(c)\phi_{22}(T) + \cdots + \phi_{11}(c_k)\phi_{22}(T^k) = r.u.T.$$

By the preceding Corollary $\phi_{22}(T^k)$ is a polynomial of degree k and leading coefficient a unit. Hence if $k > 1$, then $u.\phi_{11}(c_k) = 0$ where u is a unit, which means $\phi_{11}(c_k) = 0$. Then successively we get $\phi_{11}(c_{k-1}) = 0 = \cdots = \phi_{11}(c_2)$, and $\phi_{11}(c).uT = ruT$, which implies that $\phi_{11}(c) = r$.

Proof of the Theorem. Let $f: R[T] \to R[T]$ be an onto ring homomorphism. Then by Lemma 4, the ring homomorphism $f|R: R \to R$ is onto. Since R is a hopfian ring, $f|R$ is an isomorphism. Next we prove that f is injective. Suppose $f\begin{pmatrix} r \\ p(T) \end{pmatrix} = 0$. This means $\begin{pmatrix} \phi_{11}(r) + \phi_{12}(p(T)) \\ \phi_{22}(p(T)) \end{pmatrix} = 0$. It follows that $\phi_{22}(p(T)) = 0$. Now this last fact implies that $p(T) = 0$ because otherwise the leading coefficient a_k of $p(T)$ is nonzero; then the leading coefficient of $\phi_{22}(p(T))$ must be $u \cdot a_k$ where u is a unit in R. Since $u \cdot a_k \neq 0$, we have a contradiction. Now $p(T) = 0$ implies $\phi_{12}(p(T)) = 0$ which means $\phi_{11}(r) = 0$. But then $f(r) = \phi_{11}(r) = 0$ which means $r = 0$, as $f|R \to R$ is an isomorphism. Thus $\begin{pmatrix} r \\ p(T) \end{pmatrix} = 0$. Therefore f is a ring isomorphism, proving that $R[T]$ is hopfian.

Acknowledgement

The author is thankful to the UGC-CSIR for the award of JRF when this work was done. He is also thankful to K Varadarajan for his preprints of related papers.

References

[1] Atiyah M F and MacDonald I G, *Introduction to commutative algebra* (Addison Wesley) (1969)
[2] Deo S and Varadarajan K, Hopfian and cohopfian zero-dimensional spaces, *J. Ramanujan Math. Soc.* **9** (1994) 177–202
[3] Herstein I N, *Noncommutative rings, The carus mathematical monographs* (1973) monograph series No. 15
[4] Varadarajan K, Hopfian and cohopfian objects, *Publ. Mat.* **36** (1992) 293–317
[5] Varadarajan K, *On hopfian rings* (to appear in *Acta Math.*)

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 137–149.

Immersions in a symplectic manifold

MAHUYA DATTA

International Centre for Theoretical Physics, P.O. Box 586, 34100 Trieste, Italy
Department of Pure Mathematics, University College of Science, 35, P. Barua Sarani, Calcutta 700 019, India

MS received 2 August 1996; revised 10 February 1998

Abstract. In this paper we give a homotopy classification of symplectic isometric immersions following Gromov's h-principle theorem.

Keywords. Symplectic immersions; h-principle.

1. Introduction

Let (N, σ) be a smooth symplectic manifold and M a manifold with a closed C^∞ 2-form ω on it. A smooth immersion $f: (M, \omega) \to (N, \sigma)$ is called *symplectic isometric* (or simply *symplectic*) if f pulls back σ onto ω. The differential of f gives rise to a bundle monomorphism $F: TM \to TN$ such that $F^*\sigma = \omega$. Moreover, the underlying map $M \to N$ of F, which is f itself here, pulls back the de Rham cohomology class of σ onto that of ω. So a natural question would be if the existence of such a bundle map ensures the existence of a symplectic immersion. Gromov proves that when dim $M <$ dim N then the question has an affirmative answer. We prove here further that the space of symplectic immersions is the same up to homotopy as the space of those bundle maps. Let Symp (M, N) denote the space of symplectic immersions of M into N with C^∞ compact-open topology and Symp_0 (TM, TN) denote the associated space of bundle monomorphisms $F: TM \to TN$ with C^0 compact-open topology; i.e. F satisfies $F^*\sigma = \omega$ and its underlying (continuous) map f satisfies the cohomology condition $f^*[\sigma] = [\omega]$, where $[\sigma]$ and $[\omega]$ denote respectively the de Rham cohomology classes of σ and ω.

The main theorem may now be stated as follows.

Theorem 1.1. ([1], p. 334–335) *If dim $M <$* dim *N then the differential map d:* $\mathrm{Symp}(M, N) \to \mathrm{Symp}_0(TM, TN)$ *is a weak homotopy equivalence. In fact, symplectic immersions satisfy the C^0-dense parametric h-principle in the space of continuous maps f: $M \to N$ which pull back the cohomology class of σ onto that of ω.*

It is interesting to note that when dim $N = 2$dim M, taking ω equal to zero we obtain the following result of Lees [2].

COROLLARY 1.2

The Lagrangian immersions satisfy C^0-dense parametric h-principle.

In the next theorem we state the relative version of the above h-principle.

Theorem 1.3. *Let $F: T(\mathrm{Op}\,A) \to TN$ be a bundle monomorphism such that $F^*\sigma = \omega$, where A is a compact set in M, and let the underlying map f be a symplectic immersion on a neighbourhood of a compact set $B \subset A$. If the relative cohomology class $[f^*\sigma - \omega]$ vanishes in $H^2(A, B)$ then F can be homotoped to a symplectic immersion such that the homotopy remains constant in a neighbourhood of B.*

It should be remarked that Gromov studied in [1, § 3.4.2] a more general problem, namely the h-principle of σ-regular isometric immersions for an arbitrary closed form σ. The general theorem arises from the h-principle of some auxiliary sheaf which comes as the solution sheaf of an infinitesimally invertible differential operator, and Gromov proves this by using sophisticated machinery like the Nash–Moser implicit function theorem along with his sheaf theoretical techniques. The maps into a symplectic manifold (N, σ) automatically satisfy the σ-regularity condition because of the non-degeneracy of the 2-form σ. Moreover, we may avoid the generalized implicit function theorem. Instead, we use Moser's Theorem on stability of symplectic forms, which says that if two symplectic forms on a compact manifold are homotopic within the same cohomology class then they are isotopic. Throughout this paper we shall extensively use different consequences of this stability theorem.

The proof of the theorems is based on sheaf theoretic techniques. The sheaf of symplectic isometric immersions arises as the solution space of a closed partial differential relation [1, p. 2]. To apply the above-mentioned technique one starts with a topological solution of the diffeferential relation and embeds the manifold (M, ω) in a symplectic manifold (M', ω') such that ω' restricts to ω on M, so that the restrictions of symplectic isometric immersions $(M', \omega') \to (N, \sigma)$ to M are isometric with respect to ω. We refer to the sheaf of symplectic immersions $M' \to N$ as the extension sheaf. An important requirement for the applicability of the sheaf theoretic techniques is the existence of a microflexible extension (see § 2). However, as we shall see in Example 3.3, this is not the case here. Noting that the symplectic immersion $(M', \omega') \to (N, \sigma)$ are in 1–1 correspondence with the Lagrangian sections of the product symplectic manifold $(M' \times N, \sigma - \omega')$ and that with the cohomology condition on the topological solution we can reduce the problem to the case where $\sigma - \omega'$ is exact, we pass on to the auxiliary sheaf of exact Lagrangian sections of $M' \times N \to M'$. This sheaf has the same homotopy type as the sheaf of Lagrangian section of $M' \times N$ and, moreover, it is microflexible. Another key point is to note that Hamiltonian diffeotopies on (M', ω') act on the auxiliary sheaf and sharply move M in M' [1, p. 82]. Thus we obtain h-principle for the auxiliary sheaf and hence the h-principle for symplectic immersions. Following the proof we also obtain

COROLLARY 1.4

If the symplectic form on N is exact, namely $\sigma = \mathrm{d}\tau$, and if $\dim N = 2\dim M$, *then the space of τ-exact Lagrangian immersions satisfy (everywhere C^0-dense) h-principle.*

For any undefined term we refer to [1].

2. Brief review of the sheaf theoretic results

We now briefly describe the sheaf theoretic techniques to prove parametric h-principle. Let Φ denote the sheaf of solutions of some rth order partial differential relation $\mathscr{R} \subset J^r(M, N)$ defined for C^r-maps $M \to N$, and Ψ the sheaf of sections of the r-jet bundle

$J^r(M,N) \to M$ with images in $\mathscr{R}$. The natural topologies on $\Phi(U)$ and $\Psi(U)$ are respectively the C^r and C^0 compact open topologies.

DEFINITION 2.1

The solution sheaf Φ and the relation $\mathscr{R}$ are said to satisfy parametric h-principle if the r-jet map $j^r:\Phi \to \Psi$ is a weak homotopy equivalence.

Before proceeding further we state some general definitions and results on topological sheaves.

DEFINITION 2.2

let $\mathscr{F}$ be a topological sheaf over M and A be a compact set in M. The symbol $\mathscr{F}(A)$ will denote the space of maps which are defined over some neighbourhood of A in M; in fact it is the direct limit of the spaces $\mathscr{F}(U)$ where U runs over all the open sets containing A. A map $f: P \to \mathscr{F}(A)$ on a polyhedron P is called continuous if there exists an open set $U \supset A$ such that each f_p is defined over U and the resulting map $P \to \mathscr{F}(U)$ is continuous with respect to the given topology on $\mathscr{F}(U)$.

DEFINITION 2.3

A topological sheaf $\mathscr{F}$ over M is flexible if the restriction maps $\mathscr{F}(A) \to \mathscr{F}(B)$ are Serre fibrations for every pair of compact sets (A,B), $A \supset B$. The restriction map $\mathscr{F}(A) \to \mathscr{F}(B)$ is called a microfibration if given a continuous map $f'_0: P \to \mathscr{F}(A)$ on a polyhedron P and a homotopy f_t, $0 \leqslant t \leqslant 1$, of $f'_0|_B$ there exists an $\varepsilon > 0$ and a homotopy f'_t of f'_0 such that $f'_t|\mathrm{Op}B = f_t$ for $0 \leqslant t \leqslant \varepsilon$. If for any pair of compact sets the restriction morphism is a microfibration then the sheaf $\mathscr{F}$ is called microflexible.

DEFINITION 2.4

A sheaf homomorphism $f: \mathscr{F} \to \mathscr{G}$ is a local weak homotopy equivalence if for each $x \in M$ the homomorphism $f(x): \mathscr{F}(x) \to \mathscr{G}(x)$ is a weak homotopy equivalence.

Theorem 2.5. (sheaf homomorphism theorem [1, p. 77]) *Let $\mathscr{F}$ and $\mathscr{G}$ be two topological sheaves defined on a manifold M and let $f: \mathscr{F} \to \mathscr{G}$ be a morphism. If both sheaves are flexible and if f is a local weak homotopy equivalence then f is a weak homotopy equivalence.*

So to prove parametric h-principle for a relation $\mathscr{R}$ it suffices to show that the sheaves Φ and Ψ (as defined above) are flexible and the r-jet map j^r: $\Phi \to \Psi$ is a local weak homotopy equivalence. For any partial differential relation $\mathscr{R}$ the sheaf Ψ is always flexible [1, p. 40]. But to prove flexibility of Φ we need to impose certain extensibility conditions on $\mathscr{R}$.

Let M be embedded in a higher dimensional manifold M' and let $\mathscr{R}'$ be a relation on M'. We denote the corresponding sheaf of solutions by Φ'.

DEFINITION 2.6

Φ' is said to be an *extension* of Φ if the inclusion of M in M' induces a restriction homomorphism $\alpha: \Phi'|_M \to \Phi$; moreover, $\alpha(x)$ is a surjection for each $x \in M$.

This means that if we restrict a solution of $\mathscr{R}'$ to M we obtain a solution of $\mathscr{R}$ and moreover every local solution of $\mathscr{R}$ can be lifted to a local solution of $\mathscr{R}'$.

Now, for a pair of compact subsets (A, B) in M we define the space $\Gamma(A, B)$ of compatible pairs of solutions inside $\Phi'(B)\,\Phi(A)$. This set consists of all pairs (f', f) such that $\alpha(f') = f|\mathrm{Op}B$.

DEFINITION 2.7

The extension Φ' will be called a microextension if the obvious map $\gamma : \Phi'(A) \to \Gamma(A, B)$ is a microfibration.

Now we explain the concept of diffeotopy sharply moving a submanifold in M'. It is worth recalling that the idea contained in this definition is a key point in the Smale–Hirsch immersion Theorem.

DEFINITION 2.8

We fix a metric d on M. An open set in M will be called 'small' if it is contained in a ball of small radius. A class of diffeotopies $\mathscr{D}$ on M' is said to sharply move M in M' if given any hypersurface S lying in a small open set of M and given any positive number ε we can obtain a diffeotopy $\{\delta_t\}$ in $\mathscr{D}$ which satisfies the following conditions:

(a) δ_0 is the identity map,
(b) each δ_t is identity outside an ε-neighbourhood of S in M,
(c) $d(\delta_1(S), M) > r$ for some $r > 0$.

Gromov gives the following sufficient condition for flexibility of Φ in his main Lemma [1, p. 82] and microextension Theorem [1,. 85].

Theorem 2.9. *If Φ admits a microextension Φ' which is microflexible and if there exists a class of acting diffeotopy on Φ' which sharply moves M in M' then Φ is a flexible sheaf.*

3. Construction of an extension sheaf

Let (M, ω) and (N, σ) be as in § 1. Then the symplectic immersions $(M,\omega) \to (N, \sigma)$ correspond to the partial differential relation $\mathscr{R} \subset J^{(1)}(M, N)$ consisting of 1-jets $j^1_x f$, $x \in M$, of germs of local immersions f such that $f^*\sigma = \omega$ at x. Let Ψ denote the sheaf of bundle monomorphisms $F : TM \to TN$ which pull back the form σ onto ω. This may be identified with the sheaf of sections of the relation $\mathscr{R}$. To obtain an extension of $\mathscr{R}$, we first embed (M, ω) isometrically into a symplectic manifold (M', ω'). We start with an $F : TM \to TN$ in $\Psi(M)$ whose underlying map is $f : M \to N$ and consider the bundle f^*TN/TM over M. Without loss of generality we may assume that F is smooth. Observe that the total space of the bundle, which we denote by X, has the same dimension as N. We extend F to a bundle morphism $F' : TX|M \to TN$ such that F' maps fibres of $TX|M$ isomorphically onto the fibres of TN. Since the form $F'^*\sigma$ restricts to the closed form ω on M, it extends to a closed form ω' on some neighbourhood M' of M in X. M' may be taken to be a tubular neighbourhood of M in X so that the inclusion i: $M \to M'$ is a homotopy equivalence. Since $F'^*\sigma$ is non-degenerate so is ω'. Thus, (M, ω) is isometrically embedded in the symplectic manifold (M', ω').

We shall denote the sheaf of symplectic isometric immersions of (M, ω) in (N, σ) by $\mathscr{S}$ and that of (M', ω') in (N, σ) by $\mathscr{S}'$. Let $\mathscr{R}'$ denote the space of 1-jets of germs of symplectic immersions of (M', ω') in (N, σ) and Ψ' the sheaf of section of $\mathscr{R}'$.

PROPOSITION 3.1

$\mathcal{S}'$ is an extension of $\mathcal{S}$.

Proof. It is easy to see that the isometric embedding of (M,ω) in (M',ω') induces a morphism $\alpha:\mathcal{S}'|_M \to \mathcal{S}$. To prove that $\alpha(x):\mathcal{S}'(x)\to\mathcal{S}(x)$ is onto we start with a local symplectic immersion f at a point $x\in M$. Let $\bar{f}$ be any extension of f to a local immersion in M'. Then, since the dimension of M' is the same as the dimension of N, the form $\bar{\omega}=\bar{f}^*\sigma$ is a symplectic form. Now the two linear symplectic forms $\bar{\omega}_x$ and ω'_x defined on T_xM' coincide on the subspace T_xM. Hence there exists a linear isomorphism l of T_xM' which pulls back $\bar{\omega}_x$ onto ω'_x and keeps T_xM pointwise fixed. We consider the germ of a local map f' whose 1-jet at x equals to $j^1_x\bar{f}\circ l$ so that $j^1_x f'\in\mathcal{R}'$. By construction the jet $j^1_x f'$ projects onto $j^1_x f\in\mathcal{R}$. Moreover we may assume without loss of generality that f' extends f. So we have the following:

- $f'^*\sigma=\omega'$ at x.
- f' equals f on $U\cap M$, where U is the domain of f. Hence, pullbacks of both the forms $f'^*\sigma$ and ω' are the same.

Therefore, by the relative Poincaré Lemma, we obtain a 1-form φ on a neighbourhood, say $\tilde{U}$, of x in U such that $d\varphi=f'^*\sigma-\omega'$ and $\varphi|\tilde{U}\cap M=0$. Now, by applying Moser's theorem [3] we get a diffeomorphism δ on a neighbourhood, say U', of x in $\tilde{U}$, such that $\delta^*(f'^*\sigma)=\omega'$, $\delta|U'\cap M$ is identity, and $d\delta_x=id$. Then $f'\circ\delta$ is the required extension of f. ■

PROPOSITION 3.2

The 1-jet map $j^1:\mathcal{S}\to\Psi$ is a local weak homotopy equivalence.

Proof. The main point is to observe that an infinitesimal solution can be deformed to a local solution of the relation $\mathcal{R}$. To see this, we start with an infinitesimal solution f at x, so that $f^*\sigma=\omega$ at x. Proceeding as in the proof of the above lemma we may extend f to a map f' on a neighbourhood of x in M' such that $f'^*\sigma=\omega'$ at x. Set $f'^*\sigma=\omega''$. Since $\omega''=\omega'$ at x, therefore ω'' is symplectic on a neighbourhood of x. Applying Moser's theorem we get a local isotopy δ_t at x such that $\delta_1^*\omega''=\omega'$. Moreover, the homotopy keeps x and T_xM' pointwise fixed. Defining $\bar{f}=f'\circ\delta|_M$ on $\mathrm{Op}\,x$ we observe that $\bar{f}$ is a symplectic immersion and it is homotopic to f in the space of infinitesimal solutions of $\mathcal{R}$.

The remaining part of the proof is now a routine work in view of the above observation (and hence we omit it here). □

However, it can be seen from the following example that the extension sheaf $\mathcal{S}'$ is not microflexible.

Example 3.3. Consider the standard embedding of the closed unit disc in $\mathbb{R}^2$. If we deform it near the boundary by pushing it inside then it (the homotopy) cannot be extended symplectically on the whole of the disc.

This phenomenon may be explained as follows: If f_0 is a symplectic immersion over $\mathrm{Op}\,A$ and f_t a homotopy of f_0 such that $f_t|\mathrm{Op}B$ is a symplectic immersion, then the relative cohomology class of $f_t^*\sigma-\omega$ in $H^2(A,B)$ determines the obstruction to

extending $f_t|\mathrm{Op}B$ to Op A as symplectic immersions. If the cohomology class $[f_t^*\sigma - \omega] = 0 \in H^2(A, B)$ then there exists a smooth family of 1-forms α_t which vanishes on $\mathrm{Op}B$ and $f_t^*\sigma - \omega = d\alpha_t$. Then Moser's stability Theorem applies and we can lift $f_t|\mathrm{Op}B$ over A as symplectic immersion.

Since $\mathscr{S}'$ is not microflexible we cannot apply the sheaf theoretic techniques (described in § 2) on it. However, we shall see in the next section that there exists a topological sheaf on M' naturally associated to a subspace of the space of symplectic immersions which do satisfy microflexibility and has the same homotopy type as $\mathscr{S}'$.

4. Sheaf of exact Lagrangian sections

Throughout this section we assume that both σ and ω' are exact symplectic forms. Let p_1 and p_2 respectively denote the projections of $M' \times N$ onto the first and the second factor. The product form $p_2^*\sigma - p_1^*\omega'$ on $M' \times N$ is then an exact symplectic form. We denote it by $\sigma - \omega'$. Let τ be a 1-form such that $\sigma - \omega' = \mathrm{d}\tau$.

If $f: M' \to N$ is a symplectic isometric immersion then its graph map $g = (1, f)$: $M' \to M' \times N$ is a Lagrangian section of $(M' \times N,\ \sigma - \omega')$. Since $\sigma - \omega' = \mathrm{d}\tau$, the Lagrangian condition becomes equivalent to closeness of the form $g^*\tau$. It is easy to observe that the correspondence $f \mapsto g$ is bijective. We now construct the sheaf of exact Lagrangian sections as follows: This consists of pairs (g, φ), where $g: M' \to N$ is a section of the product bundle such that map $f = p_2 \circ g: M' \to N$ is an immersion, and φ is a function on M' satisfying $g^*\tau = \mathrm{d}\varphi$. (Such a g is called a τ-exact Lagrangian immersion.) We denote the sheaf of such pairs by $\mathscr{E}'$ and call it the sheaf of τ-exact Lagrangian sections. Observe that $\mathscr{S}'$ and $\mathscr{E}'$ are locally homotopically equivalent since the germ of a Lagrangian section at a point denotes a germ of an exact Lagrangian section; moreover the space of primitives φ for a τ-exact Lagrangian section g (meaning that φ satisfies the relation $g^*\tau = \mathrm{d}\varphi$) is isomorphic to $\mathbb{R}$. Consequently, the sheaf of sections corresponding to the relation, of which $\mathscr{E}'$ is the solution sheaf, has the same homotopy type as Ψ'. We now prove

PROPOSITION 4.1

The sheaf $\mathscr{E}'$ of τ-eact Lagrangian sections is microflexible.

Proof. Let (A, B) be a pair of compact sets in M'. Let g' be a τ-exact Lagrangian section over a A (meaning that it is defined on a neighbourhood of A) such that $g'^*\tau = \mathrm{d}\varphi'$ for a 0-form φ', and (g_t, φ_t) a homotopy of $(g', \varphi')|\mathrm{Op}B$ in $\mathscr{E}'$.

We first prove the following simple lemma.

Lemma 4.2 *Let g_t be a homotopy of τ-exact Lagrangian sections. If g_0 is also a τ'-exact Lagrangian section for some 1-form τ' on $M' \times N$ satisfying $\sigma - \omega' = \mathrm{d}\tau'$, then every g_t is a τ'-exact Lagrangian section.*

Proof. Two such forms τ and τ' differ by a closed 1-form c on M'. So, we have the following relation

$$g_t^*\tau' = g_t^*\tau + g_t^*c$$

for every t. Then, by hypothesis, g_0^*c is an exact form. Since c is closed, g_t^*c is also exact. Consequently $g_t^*\tau'$ is exact for each t. ■

Now by the standard theory of Lagrangian submanifolds [3], there exists a neighbourhood W of the Lagrangian submanifold $L' = \operatorname{Im} g'$ such that $(W, \mathrm{d}\tau)$ is symplectomorphic to a neighbourhood of the zero section $Z_{L'}$ in the cotangent bundle $(T^*L', \mathrm{d}\theta_{L'})$ with the standard symplectic form $\mathrm{d}\theta_{L'}$ on it. Under the above identification, a Lagrangian section of $M' \times N$ (inside W) which is a small deformation of g' corresponds to a C^0-small closed form on L'; moreover, a $\tau' = \delta^*\theta_{L'}$-exact Lagrangian corresponds to an exact form on L'. Clearly the sheaf of exact 1-forms is microflexible. Hence we can obtain lifts g'_t *of* g_t (for t small enough) which are τ'-exact Lagrangian sections. By the Lemma above they are also τ-exact. Moreover, for small t, the underlying maps will be immersions on Op A. Now, we can choose a homotopy φ'_t on Op A such that $g'^*_t\tau = \mathrm{d}\varphi'_t$. On Op B, we have $\mathrm{d}\varphi'_t = \mathrm{d}\varphi_t$. Hence $\varphi'_t - \varphi_t = c_t$, where c_t is a closed 0-form, that is a constant. So we may replace φ'_t by $\varphi'_t - c_t$. The homotopy $(g'_t, \varphi'_t - c_t)$ is the required lift. ■

We shall now describe a class of diffeotopies which would act on the sheaf $\mathscr{E}'$ and at the same time sharply move a submanifold of M' of positive codimension. Since ω' is symplectic we have an isomorphism $I_{\omega'}: \chi(M') \to \Lambda^1(M')$ from the space of vector fields $\chi(M')$ onto the space of 1-forms $\Lambda^1(M')$. A C^∞ diffeotopy δ_t of M' is called exact if δ_0 is identity and if $\delta'_t = \mathrm{d}\delta_t/\mathrm{d}t$ is a Hamiltonian vector field for each t. So we can write $\delta'_t \cdot \omega' (= I_{\omega'}(\delta'_t)) = \mathrm{d}\alpha_t$ for some smooth family of exact 1-forms $\mathrm{d}\alpha_t$ on M'. If α_t can be chosen to be identically zero on the maximal open subset where δ_t is constant, then such a diffeotopy is called a strictly exact diffeotopy.

PROPOSITION 4.3

The (strictly) exact diffeotopies of M' act on the sheaf $\mathscr{E}'$.

Proof. Let δ_t be a strictly exact diffeotopy on M'. We define a diffeotopy $\bar{\delta}_t$ on $M' \times N$ by $\bar{\delta}_t(x, y) = (\delta_t(x), y)$, where $x \in M'$ and $y \in N$. It follows that $\bar{\delta}'_t \cdot (\sigma - \omega')$ is exact for each t. Let α_t be a smooth family of 0-forms on $M' \times N$ satisfying $\bar{\delta}'_t \cdot (\sigma - \omega') = \mathrm{d}\alpha_t$. Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\delta}^*_t \tau) = \mathscr{L}_{\bar{\delta}'_t}\tau &= \bar{\delta}^*_t(\mathrm{d}(\bar{\delta}'_t \cdot \tau) + \bar{\delta}'_t \cdot \mathrm{d}\tau)\\ &= \bar{\delta}^*_t(\mathrm{d}(\bar{\delta}'_t \cdot \tau) + \bar{\delta}'_t \cdot \sigma - \bar{\delta}'_t \cdot \omega')\\ &= \bar{\delta}^*_t(\mathrm{d}(\bar{\delta}'_t \cdot \tau) + \mathrm{d}\alpha_t)\\ &= \mathrm{d}\bar{\delta}^*_t((\bar{\delta}'_t \cdot \tau) + \alpha_t).\end{aligned}$$

If we define $\varphi_t = \int_0^t \bar{\delta}^*_t(\bar{\delta}'_t \cdot \tau + \alpha_t)\,\mathrm{d}t$ then $\bar{\delta}^*_t \tau = \tau + \mathrm{d}\varphi_t$. Now we are in a position to define the action. For $(g, \varphi) \in \mathscr{E}'$ and δ_t as above, we set

$$\delta^*_t(g, \varphi) = (\delta^*_t g, (\delta_t^{-1})^*(\varphi + g^*\varphi_t)),$$

where $\delta^*_t g = \bar{\delta}_t \circ g \circ \delta_t^{-1}$. ■

PROPOSITION 4.4

The strictly exact diffeotopies of the symplectic manifold (M', ω') sharply move M in M'.

Proof. (Gromov) To move a closed hypersurface S lying in a small open set U of M we start with a vector $\partial_0 \in T_{x_0}(M')$ transversal to U in M'. This ∂_0 extends to an exact field

$\partial = I_{\omega'}^{-1}(\mathrm{d}H)$ which is transversal to U, since U is chosen small. In order to make the corresponding exact isotopy δ_t sharply move S, we take the union $S_\varepsilon = \cup_t \delta_t(S) \in M'$ over $t \in [0, \varepsilon]$ and then multiply the Hamiltonian H by a properly chosen C^∞ function a on U which vanishes outside an arbitrarily small neighbourhood of Op S_ε and which equals r in a smaller neighbourhood of S_ε. The diffeotopy corresponding to the field $I_{\omega'}(\mathrm{d}(aH))$ can move S as sharply as we want; the required sharpness can be obtained with a proper choice of ε. ∎

Now applying the Main lemma of Gromov [1, p. 82] we may conclude as follows.

PROPOSITION 4.5

The sheaf $\mathscr{E}'|_M$ is flexible.

It then follows from the sheaf homomorphism Theorem that $\mathscr{E}'|_M$ satisfies parametric h-principle.

Let $\mathscr{E}$ be the sheaf of pairs (g, φ) on M, where $g: M \to M' \times N$ is a section such that its composition with the projection map p_2 is an immersion and φ is a function on M satisfying the relation $g^*\tau = \mathrm{d}\varphi$. To descend h-principle from $\mathscr{E}'|_M$ to $\mathscr{E}$ we observe the following.

PROPOSITION 4.6

$\mathscr{E}'$ is a microextension of $\mathscr{E}$.

Proof. From Proposition 3.1 and the discussion preceding Proposition 4.1 it follows that $\mathscr{E}'$ is an extension of $\mathscr{E}$. To prove that $\mathscr{E}'$ is a microextension of $\mathscr{E}$ we consider a lifting problem

$$\begin{array}{ccc} P \times \{0\} & \xrightarrow{(G'_0, \psi'_0)} & \mathscr{E}'(A) \\ i \big\downarrow & & \big\downarrow \gamma \\ P \times I & \xrightarrow[(g', \varphi'), (g, \varphi)]{} & \Gamma(A, B) \end{array}$$

where $\alpha \circ (G'_0, \psi'_0) = (g_0, \varphi_0)$ and $(G'_0, \psi'_0) | \mathrm{Op}B = (g'_0, \varphi'_0)$ and where $\Gamma(A, B)$ is a subset of $\mathscr{E}'(B) \times \mathscr{E}(A)$ consisting of compatible solutions as defined in § 2. (To avoid too many symbols we assume P to be a point and denote $g(t)$ by g_t and so on.) We shall denote the underlying maps of G'_0, g'_t and g_t by F'_0, f'_t and f_t. Since they are immersions (which correspond to an open differential relation), we can obtain a lift of the corresponding microextension problem for immersions. Let us denote the lift by F_t, where $0 \leqslant t \leqslant \varepsilon$ for some positive number $\varepsilon \leqslant 1$. Now each F_t being an immersion between equidimensional spaces, pulls back σ onto a symplectic form on a neighbourhood of A. Let us set $F_t^*\sigma = \omega'_t$. We denote the corresponding graph map by G_t. Then we have the relation $F_t^*\sigma - \omega' = \mathrm{d}G_t^*\tau$. On the other hand we obtain a homotopy ψ'_t of ψ'_0 such that ψ'_t coincides with φ_t and φ'_t on the relevant spaces. The 1-form α_t defined by $\alpha_t = G_t^*\tau - \mathrm{d}\psi'_t$ satisfies the following.

(a) $\alpha_0 = 0$
(b) α_t vanishes on some open neighbourhood of A in M,
(c) α_t vanishes on an open neighbourhood of B in M'
(d) $F_t^*\sigma - \omega' = d\alpha_t$.

Consider the vector fields $X_t = I_{\omega_t'}^{-1}(d\alpha_t/dt)$. The vector fields vanish on $\mathrm{Op}_M A$ as well as on $\mathrm{Op}_{M'} B$. Hence it can be integrated on a neighbourhood of A in M' to obtain a family of diffeomorphisms $\{\delta_t; 0 \leqslant t \leqslant \tilde{\varepsilon}\}$ for some $\tilde{\varepsilon} \leqslant \varepsilon$ such that

(e) δ_0 is identity on $\mathrm{Op}_{M'} A$,
(f) $\delta_t | \mathrm{Op}_M A = id$,
(g) $\delta_t | \mathrm{Op}_{M'} B = id$,
(h) $\delta_t^* \omega_t' = \omega'$.

The required partial lift of the original lifting homotopy problem can now be given by the graph map of $F_t' = F_t \circ \delta_t$. In fact, since F_t' is a symplectic immersion $G_t'^* \tau$ is closed. On the other hand, $i: M \to M'$ induces an isomorphism $i^*: H_{deR}^2(M') \to H_{deR}^2(M)$, and we know from our initial data that $i^* G_t'^* \tau$ is exact. Hence, $G_t'^* \tau$ is also exact. It is now a trivial matter to fix ψ_t'. □

With the microextension Theorem of Gromov [1, p. 85] we now observe from Proposition 4.5 and 4.6 that the sheaf $\mathscr{E}$ is flexible. We have already proved the local h-principle in Proposition 3.2. So again appealing to the sheaf homomorphism Theorem we may conclude that $\mathscr{E}$ satisfies parametric h-principle.

Corollary 1.4 is now an easy consequence of the above h-principle. If the symplectic form on N is exact, namely $\sigma = d\tau$, and if $\dim N = 2 \dim M$, then for $\omega = 0$ the space $\mathscr{E}(M)$ corresponds to the space of τ-exact Lagrangian immersions. This proves that the space of τ-exact Lagrangian immersions satisfy C^0-dense h-principle.

We now prove as follows.

PROPOSITION 4.7

$\mathscr{E}(M)$ has the same homotopy type as the space $\mathscr{S}(M)$ of symplectic isometric immersions.
To prove this we need the following.

Lemma 4.8 *Let (M,ω) and (N,σ) be two symplectic manifolds with exact symplectic forms and let M be compact. Let g be a section of $M \times N \to M$ such that the underlying map $f = p_2 \circ g: M \to N$ is a symplectic immersion. Then f can be homotoped to a symplectic immersion f_1 such that $g_1 = (1, f_1)$ is a τ-exact Lagrangian immersion. In fact, $(p_2)_*: \mathscr{E}(M) \to \mathscr{S}(M)$ induces surjective maps at each homotopy level.*

Proof. Since both σ and τ are exact, there exists a 1-form τ on $M \times N$ such that $p_2^*\sigma - p_1^*\omega$ equals $d\tau$. Therefore, if f is a symplectic immersion, then $g^*\tau$ is closed. We denote the form $g^*\tau$ by c. Since ω is a symplectic form there exists a unique vector field ∂ defined by $I_\omega^{-1}(-c)$. Now, M being compact, the vector field ∂ can be integrated on M to obtain an isotopy δ_t so that $\delta_0 = id$ and $(d/dt)\delta_t = \partial$.

Define $f_t = f \circ \delta_t$ for $t \in I$. We first prove that each $f_t: M \to N$ is a symplectic immersion. This will imply that $g_t = (1, f_t)$ is a Lagrangian section of $M \times N$. To prove this we observe that

$$\frac{d}{dt}\delta_t^*\omega = \delta_t^*\{\partial \cdot d\omega + d(\partial \cdot \omega)\} = \delta_t^*(0 + d(-c)) = 0.$$

Therefore, $\delta_t^*\omega = \delta_0^*\omega = \omega$. And hence, each f_t is a symplectic immersion with respect to σ and ω.

Next we prove that the 1-form c is invariant under each δ_t, that is, $\delta_t^* c = c$ for every $t \in I$. In fact, differentiating $\delta_t^* c$ with respect to t we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta_t^* c = \delta_t^*(\partial\cdot \mathrm{d}c + \mathrm{d}(\partial\cdot c)).$$

Since the form c is closed and satisfies the relation $c = -\partial\cdot\omega$, the right hand side is equal to zero. Thus, $\delta_t^* c = \delta_0^* c = c$.

Finally we will show that $g_1^*(\tau)$ is exact, that is, g_1 is a τ-exact Lagrangian immersion. Let $\bar{\delta}_t$ denote the map $M \to M \times M$ defined by $\bar{\delta}_t(x) = (x, \delta_t(x))$. Then $g_t = (1 \times f)\circ\bar{\delta}_t$.

If we set $\tau' = (1 \times f)^*\tau$ then

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t^*\tau = \frac{\mathrm{d}}{\mathrm{d}t}\bar{\delta}_t^*\tau' = \bar{\delta}_t^*\{\bar{\partial}\cdot \mathrm{d}\tau' + \mathrm{d}(\bar{\partial}\cdot\tau')\}, \tag{1}$$

where $\bar{\partial} = (\mathrm{d}/\mathrm{d}t)\bar{\delta}_t = (0, \partial)$. Now, it is easy to see that $(1 \times f)^*\mathrm{d}\tau = p_2^*\omega - p_1^*\omega$. Hence it follows that,

$$\begin{aligned} \bar{\delta}_t^*(\bar{\partial}\cdot \mathrm{d}\tau')(V) &= \mathrm{d}\tau'((0,\partial),(V, \mathrm{d}\delta_t(V))) \\ &= \omega(\partial, \mathrm{d}\delta_t(V)) - \omega(0, V) \\ &= \delta_t^*(\partial\cdot\omega)(V) \\ &= -\delta_t^*(c)(V) = -c(V). \end{aligned}$$

Equation (1) then reduces to the following:

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t^*\tau = -c + \bar{\delta}_t^*\mathrm{d}(\bar{\partial}\cdot\tau') = -c + \mathrm{d}\alpha_t,$$

where $\alpha_t = \bar{\delta}_t^*(\bar{\partial}\cdot\tau')$.

Integrating the above relation with respect to t we get $g_t^*\tau - c = -tc + \mathrm{d}\beta_t$, where $\beta_t = \int_0^t \alpha_t$. Substituting $t = 1$ we see that $g_1^*\tau$ is exact. Thus we prove that p_2 induces a surjective map $\pi_0(\mathscr{E}(M)) \to \pi_0(\mathscr{S}(M))$. The above arguments can easily be extended for a family of maps parametrized by a compact polyhedron to complete the proof of the lemma. □

Remark. If f in the above lemma is such that the 1-form c equals $\mathrm{d}\varphi$ on a neighbourhood of a compact set A, then taking an extension $\varphi': M \to \mathbb{R}$ of $\varphi|\mathrm{Op}A$ and replacing c by $c - \mathrm{d}\varphi'$ in the proof, we can homotope f to an f_1 as above, keeping f fixed on Op A. Also, starting with a family of symplectic immersions f_s parametrized by a compact polyhedron P, we can deform it to a desired one, so that if $g_s = (1, f_s)$ is an exact Lagrangian for s in the boundary of P then f_s remains fixed under the deformation.

Proof of the Proposition. Passing to the extension sheaf we consider the following sequence of maps: $\mathscr{E}'|_M \xrightarrow{(p_2)_*} \mathscr{S}'|_M \xrightarrow{J^1} \Psi'|_M$. The C^0-dense parametric h-principle for $\mathscr{E}'$ says that the composition is a weak homotopy equivalence. Hence p_2 induces injective maps in the homotopy level.

On the other hand, noting the fact that the closed 2-form ω' on M' is symplectic it follows from the lemma above that p_2 induces surjective maps between homotopy

groups of the spaces $\mathscr{E}'(A)$ and $\mathscr{S}'(A)$ for every compact subset A of M. Thus, $(p_2)_*:\pi_i(\mathscr{E}(A))\to\pi_i(\mathscr{S}(A))$ is an isomorphism for every compact set $A\subset M$. It also follows from the above remark that, for a pair of compact sets (A, B) in M, the fibres of the restriction maps $\mathscr{E}'(A)\to\mathscr{E}'(B)$ and $\mathscr{S}'(A)\to\mathscr{S}'(B)$ are of the same weak homotopy type.

Since M can be covered by an increasing sequence of compact sets in M, we can conclude that $(p_2)_*:\mathscr{E}'|_M\to\mathscr{S}'|_M$ is a weak homotopy equivalence.

Now, proceeding as in Proposition 4.6 we may observe that both restriction maps $\mathscr{S}'(M)\to\mathscr{S}(M)$ and $\mathscr{E}'(M)\to\mathscr{E}(M)$ are fibrations. Moreover, for any $g\in\mathscr{E}$, the fibres in $\mathscr{E}'(M)$ and $\mathscr{S}'(M)$ over g and $p_2\circ g$ respectively are homotopically equivalent. Since we have proved above that $\mathscr{E}'(M)$ and $\mathscr{S}'(M)$ are of the same weak homotopy type, we conclude through homotopy exact sequence of fibrations that $\mathscr{E}(M)$ and $\mathscr{S}(M)$ are also of the same weak homotopy type. This completes the proof of the proposition. □

This leads us to the following intermediate theorem.

Theorem 4.9. *If the differential forms σ and ω are exact then the space of symplectic immersions of M into N satisfies C^0 dense parametric h-principle.*

As an application to Theorem 4.8 we may look at the question of isometric immersibility of a symplectic manifold in the Euclidean symplectic manifold $\mathbb{R}^{2n}$ with its canonical symplectic structure $\omega_0=\Sigma_{i=1}^n \mathrm{d}x_i\wedge \mathrm{d}y_i$. One simple observation is that a symplectic manifold with a non-exact symplectic form cannot admit such an immersion in $\mathbf{R}^{2n}$. So an appropriate question would be if an arbitrary manifold M with an exact symplectic form ω on it admits a symplectic immersion in $\mathbb{R}^{2n}$ for some values of n.

Example 4.10. If $(M,\mathrm{d}\tau)$ is a symplectic manifold of dimension $2m$ then it admits a symplectic immersion in $(\mathbb{R}^{2n},\omega_0)$ for $n\geqslant 2m$.

To prove the existence of a symplectic immersion, it is enough to produce a section of the complex Stiefel bundle over M with fibre $\mathscr{F}_m(\mathbb{C}^n)$, which has the same homotopy type as the space of symplectic isometric bundle maps $TM\to T\mathbb{R}^{2n}$. In fact, the space of all symplectic linear maps $\mathbb{R}^{2m}\to\mathbb{R}^{2n}$ between the euclidean symplectic spaces has the homotopy type of the space of complex linear isometries $\mathbb{C}^m\to\mathbb{C}^n$, that is, the space $\mathscr{F}_m(\mathbb{C}^n)$ of all complex m-frames in $\mathbb{C}^n$. The obstruction to the existence of such a section lies in the cohomology group $H^{2m}(M,\pi_{2m-1}(\mathscr{F}_m(\mathbb{C}^n)))$. From the standard homotopy theory it follows that $\mathscr{F}_m(\mathbb{C}^n)$ is $2(n-m)$ connected. Thus, M admits a symplectic immersion in $\mathbb{R}^{2n}$ if $2m-1\leqslant 2(n-m)$ i.e., if $2m\leqslant n$.

5. Proof of the main theorem

Let us now go back to the general case where $\sigma-\omega$ is not necessarily exact on $M\times N$. However, if $f:M\to N$ is a continuous map such that $f^*[\sigma]=[\omega]$ then f can be extended to a map $f':M'\to N$ such that $f'^*[\sigma]=[\omega']$. Then, in some neighbourhood W of graph f, there exists a 1-form τ such that $\sigma-\omega'=\mathrm{d}\tau$. We shall denote by $\mathscr{S}_W$ the sheaf of symplectic immersions $M\to N$ whose graphs lie in W. Then from Theorem 4.8 it follows that $\mathscr{S}_W$ satisfies parametric h-principle. We now come to the proof of Theorem 1.1.

Proof of Theorem 1.1 In view of the above discussion it remains only to prove that d induces injective maps at each homotopy level; namely, $d_*:\pi_i(\mathcal{S}(M))\to\pi_i(\mathrm{Symp}_0(TM,TN))$ is injective for each integer i. Let f_0 and f_1 be two symplectic immersions on M such that their differentials $\mathrm{d}f_0$ and $\mathrm{d}f_1$ are homotopic in $\mathrm{Symp}_0(TM,TN)$; that is, there exists a homotopy $F_t:TM\to TN$ of symplectic bundle maps between $\mathrm{d}f_0$ and $\mathrm{d}f_1$ such the underlying maps $f_t:M\to N$ satisfy $f_t^*[\sigma]=[\omega]$. For each t we can choose a neighbourhood W_t of graph f_t on which $\sigma-\omega$ is exact. Let $\mathcal{S}_t$ denote the sheaf of those symplectic immersions whose graphs lie in W_t, and let Ψ_t be defined as in § 3 corresponding to $\mathcal{S}_t$. We can cover the set $\cup_t f_t(M)$ by finite by many such W_t's such that any two consecutive ones (ordered by the real number) intersect in a set which contains completely the graph of some f_t. Without any loss of generality we may assume that the neighbourhoods $\{W_0, W_1\}$ have this property. Let for some T, the graph of f_T lies in $W_0\cap W_1$. Then by h-principle of the sheaf $\mathcal{S}_{W_0\cap W_1}$ we obtain a symplectic immersion f, C^0-close to f_T, such that $\mathrm{d}f$ and F_T are homotopic within Ψ_{W_0}. In fact, the homotopy lies in $\mathrm{Symp}_0\,(TM,TN)$ and the underlying maps of the homotopy have their graphs in $W_0\cap W_1$. Then applying parametric h-principle for $\mathcal{S}_0$ we conclude that f and f_0 are homotopic within the space $\mathcal{S}_0$. On the other hand f is homotopic to f_1 within the space $\mathcal{S}_1$. Joining these two homotopies we obtain a homotopy between f_0 and f_1 in the space of symplectic immersions. This proves that the differential d induces an isomorphism between the homotopy groups at the zero level.

Working with a family of such maps parametrized by spheres S^i, we can similarly prove the isomorphism between the higher homotopy groups of the relevant spaces which gives the desired h principle. □

We now prove the relative or the extension version of h-principle for symplectic immersions.

Proof of Theorem 1.3. Let F be as in the statement of Theorem 1.3. Therefore, by hypothesis, the underlying map f is symplectic over Op B and the de Rham cohomology class $[f^*\sigma-\omega]$ vanishes in $H^2(A,B)$. Hence we can find a 1-form φ on M which vanishes on a neighbourhood of B and satisfies the condition $f^*\sigma-\omega=\mathrm{d}\varphi$. Let $g=(1,f)$ denote the graph map of f and let Y be a tubular neighbourhood of Im g with a retraction $r:Y\to\mathrm{Im}\ g$ so that $r\circ g=g$. Therefore, the composition $p_1\circ r\circ g$ is the identity map on M. Hence we can find a 1-form τ on Y, namely $\tau=(p_1\circ r)^*\varphi$, such that $g^*\tau=\varphi$. Thus, the g-pullbacks of the closed forms $(\sigma-\omega)$ and $\mathrm{d}\tau$ are the same and so, by the relative Poincaré Lemma, $\sigma-\omega$ equals to the exact form $\mathrm{d}(\tau+\tau')$ on a neighbourhood W of Im g, where τ' vanishes on the graph of f. Moreover, $g^*(\tau+\tau')=\varphi$ vanishes on Op B so that $(g,\varphi)|\mathrm{Op}B$ is in $\mathcal{E}_W(B)$, where $\mathcal{E}_W$ denotes the sheaf of $\tau+\tau'$-exact Lagrangian sections with images in W. Now consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{E}_W(A) & \longrightarrow & \Psi_W(A) \\ \downarrow & & \downarrow \\ \mathcal{E}_W(B) & \longrightarrow & \Psi_W(B) \end{array}$$

where the horizontal arrows are weak homotopy equivalences and the vertical ones are fibrations. Hence the fibres over $g|_B$ and $\mathrm{d}f|_{TB}$ are also weak homotopy equivalent. The theorem follows as F is in the fibre over $\mathrm{d}f|_{TB}$. □

Acknowledgements

The author would like to thank Prof. M Gromov for his many useful suggestions and comments. The author is also thankful to the referee for critically reviewing the paper.

References

[1] Gromov M, Partial Differential Relations, *Ergebnisse der Math.* (Springer-Verlag) (1986) Vol. 9

[2] Lees J On the Classification of Lagrange Immersions, *Duke Math. J.* **43** (1976) 217–224

[3] Weinstein A, Symplectic Manifolds and their Lagrangian Submanifolds, *Advances in Math.* **6** (1971) 329–346

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 151–162.

An axiomatic approach to equivariant cohomology theories

AMIYA MUKHERJEE and ANIRUDDHA C NAOLEKAR
Stat-Math Division, Indian Statistical Institute, 203, B.T. Road, Calcutta 700 035, India

MS received 23 July 1997; revised 24 January 1998

Abstract. This paper presents a translation of a theorem of Cartan into an equivariant setting. This work is largely based on the study of the homotopical algebra in the sense of Quillen of the category of simplicial objects over the category of rational O_G-vector spaces. The application is a solution to the equivariant commutative cochain problem. This solution is slightly better than the solution obtained earlier by Triantafillou in that the transformation group G need not be finite.

Keywords. Cohomology of G-simplicial sets; simplicial differential graded algebra; Cartan cohomology; closed model category; O_G-Eilenberg–MacLane complex.

1. Introduction

In [2] Cartan introduced the notion of a 'cohomology theory' and used it to generalize Sullivan's theory of rational de Rham complex to simplicial cochain algebra. Recall that a simplicial differential graded algebra A over a ring R with 1 is a simplicial object in the category **DGA/R** of differential graded algebras over R so that for each $p \geqslant 0$ we have a differential graded algebra

$$(A_p^*, \delta): A_p^0 \xrightarrow{\delta} A_p^1 \xrightarrow{\delta} A_p^2 \longrightarrow \dots$$

together with face and degeneracy maps $d_i: A_p^* \to A_{p-1}^*$ and $s_i: A_p^* \to A_{p+1}^*$ which are homomorphisms of differential graded algebras satisfying the usual simplicial identities. Then a cohomology theory in the sense of Cartan is a simplicial differential graded algebra A over R such that (1) each cochain complex (A_p^*, δ) is exact, and $Z^0 A = \ker(\delta: A_*^0 \to A_*^1)$ is a simplicially trivial R-algebra (simplicially trivial means all the d_i and s_i are isomorphisms), (2) the homotopy groups $\pi_i(A_*^n)$ of the simplicial set $A_*^n = \{A_p^n\}_{p \geqslant 0}$ are trivial whenever $i, n \geqslant 0$.

A cohomology theory A determines a contravariant functor from the category $\mathcal{S}$ of simplicial sets to **DGA/R** by sending $K \in \mathcal{S}$ to the differential graded algebra $A(K) = \{\mathrm{Hom}(K, A_*^n)\}_{n \geqslant 0}$, where $\mathrm{Hom}(K, A_*^n)$ is the R-module of simplicial maps $K \to A_*^n$, and differential and multiplication are induced from those of A. Then the theorem of Cartan is as follows:

Theorem 1.1. *If A is a cohomology theory, then there is a natural isomorphism*

$$H^*(A(K)) \cong H^*(K; R(A)),$$

on simplicial sets K, where $R(A)$ is the R-module $(Z^0 A)_0$.

The present paper is concerned with a generalization of this theorem in an equivariant set-up. Theorem 1.1 has its origin in the commutative cochain problem which was posed by Thom in 1957. A solution to this problem entails the construction of a contravariant functor A: **TOP** $\to$ **CDGA/R** (the category of commutative

differential graded algebras over R) so as to yield a de Rham type theorem which asserts that there is an isomorphism

$$H^*(A(X)) \cong H^*(X; R)$$

for every topological space X, where the cohomology on the right is the singular cohomology (note that without the commutativity requirement a cochain problem does not exist since the usual construction of cochains renders an automatic solution to it). For example, the classical de Rham theorem provides a solution for the subcategory of smooth manifolds where $A(X)$ is the commutative differential graded algebra over $\mathbb{R}$ of smooth differential forms on a manifold X. On the other hand, the commutative cochain problem has no solution over the integers, the cohomology operations (such as Steenrod squares, etc.) being the obstructions. This difficulty does not arise over the rationals $\mathbb{Q}$, or any field containing $\mathbb{Q}$. In [9], Quillen solved the rational commutative cochain problem in an abstract setting. Then Sullivan [10] gave another proof using his theory of minimal models and the de Rham complex $A(K)$ of rational polynomial forms on a simplicial set K. An independent proof, which is based on an earlier proof by Thom in the real case, was given by Swan [11] when the coefficient ring R is a field of characteristic zero. Finally, Cartan [2] formulated the main ideas of Swan in the form of axioms for a chomology theory, and proved Theorem 1.1 from which one can recover Sullivan's PL de Rham theorem for a suitable choice of cohomology theory (see [2], Example 3). The main features of both [11] and [2] is that they avoid integration of forms which is standard to proofs of de Rham type theorems.

As an application of our main theorem (see Theorem 1.4 below), we propose a solution to the commutative cochain problem for the equivariant cohomology of a G-space in the spirit of Cartan's method. This problem has already been solved by Triantafillou [12], Theorem 4.9, using Sullivan's method when G is a finite group. Triantafillou needed the finiteness condition for the use of Bredon cohomology and, more importantly, for the construction of certain projective rational coefficient system (see [12], p. 515). In our formulation we do not require G to be finite. Throughout we let G be a discrete group, and we continue to suppose that R is a commutative ring with 1. Let $\mathbf{O}_G$ be the category of canonical orbits whose objects are left coset spaces G/H and morphisms are equivariant maps $\hat{g}: G/H \to G/H'$, corresponding to subconjugacy relations $g^{-1}Hg \subseteq H'$. Let $\mathbf{RO}_G$**-mod** denote the category of $\mathbf{O}_G$-R-modules, which are contravariant functors from $\mathbf{Q}_G$ to the category **R-mod** of R-modules.

In § 2 we construct for a G-simplicial set K and a coefficient system $\lambda \in \mathbf{RO}_G$**-mod** an equivariant cohomology $H^*_G(K; \lambda)$, which is a simplicial version of the Bredon–Illman cohomology (see [1, 6]) in the sense that if X is a G-space, then the Bredon–Illman cohomology groups $\hat{H}^*_G(X; \lambda)$ are isomorphic to $H^*_G(SX; \lambda)$, where SX is the singular G-simplicial set associated to X.

DEFINITION 1.2

Let $\mathscr{C}_R$ be the category of cohomology theories over R in the sense of Cartan. Then a G-cohomology theory over R is a contravariant functor $A: \mathbf{O}_G \to \mathscr{C}_R$.

Thus for each $G/H \in \mathbf{O}_G$, $A(G/H)$ is a cohomology theory over R, and we have therefore a sequence of contravariant functors $A^n: \mathbf{O}_G \to$ **sR-mod** (the category of simplicial R-modules), $n \geqslant 0$, defined by $A^n(G/H) = A(G/H)^n_*$. We may therefore think of the functors A^n as simplicial objects in the abelian category $\mathbf{RO}_G$**-mod.** Given

a G-simplicial set K, define a contravariant functor $\Phi K:\mathbf{O}_G \to \mathbf{sR\text{-}mod}$ by $\Phi K(G/H) = \mathrm{R}K^H$, whose set of p-simplexes $(RK^H)_p$ is the free R-module RK_p^H with basis the set of p-simplexes K_p^H of K^H. Again, we may think of ΦK as a simplicial object in $\mathbf{RO_G\text{-}mod}$.

DEFINITION 1.3

Let A be a G-cohomology theory over R and $G\mathscr{S}$ the category of G-simplicial sets. We define a contravariant function $A_G: G\mathscr{S} \to \mathbf{DGA/R}$ by

$$A_G(K) = \{\mathrm{Hom}(\Phi K, A^n)\}_{n \geqslant 0},$$

where $\mathrm{Hom}(\Phi K, A^n)$ denotes the R-module of simplicial maps $\Phi K \to A^n$ in the category $\mathbf{sRO_G\text{-}mod}$ of simplicial objects in $\mathbf{RO_G\text{-}mod}$.

Also, we define $\lambda_A \in \mathbf{RO_G\text{-}mod}$ by $\lambda_A(G/H) = (Z^0 A(G/H))_0$. Then our main theorem is the following

Theorem 1.4. *Let A be a G-cohomology theory over R. Then there is a natural isomorphism*

$$H_G^*(K; \lambda_A) \cong H^*(A_G(K)),$$

on G-simplicial sets K.

Given a G-cohomology theory A, we may define $\hat{A}_G$: **G-spaces** $\to$ $\mathbf{DGA/R}$ by $\hat{A}_G(X) = A_G(SX)$. Then we shall also prove the following

Theorem 1.5. *Let $\lambda \in$ $\mathbf{RO_G\text{-}mod}$. Then there exists a G-cohomology theory A with $\lambda_A = \lambda$ such that*

$$\hat{H}_G^*(X; \lambda) \cong H^*(\hat{A}_G(X)),$$

for every G-space X.

Here are some examples which will illustrate the ideas involved in the development of the proposed G-cohomology theory.

Example 1.6. Take $R = \mathbb{R}$, the field of real numbers, and Ω as the simplicial differential graded algebra where $\Omega_p^* = \Omega^*(\Delta^p)$ is the differential graded algebra of smooth differential forms on the standard p-simplex Δ^p in $\mathbb{R}^{p+1}$. Then Ω is a cohomology theory over $\mathbb{R}$ in the sense of Cartan with $R(\Omega) = \mathbb{R}$, and the constant functor $A: \mathbf{O}_G \to \mathscr{C}_{\mathbb{R}}$ defined by $A(G/H) = \Omega$ is a G-cohomology theory. Theorem 1.4 says that $H^*(A_G(K))$ is isomorphic to the Bredon–Illman cohomology $H_G^*(K; \lambda)$, where λ is the constant coefficient system $\lambda(G/H) = \mathbb{R}$, for every G-simplicial set K.

This result may be called the equivariant de Rham theorem, because it computes the G-cohomology of K from the de Rham complexes of various fixed point sets K^H.

Example 1.7. For an R-module M, consider the simplicial differential graded algebra $C(M)$ where $C_p^*(M) = \oplus_{n \geqslant 0} C^n(\Delta[p]; M)$ is the differential graded algebra of cochains of the contractible simplicial set $\Delta[p]$ with values in M, and a coefficient system $\lambda: \mathbf{O}_G \to \mathbf{R\text{-}mod}$. Define a contravariant functor $A: \mathbf{O}_G \to \mathscr{C}_R$ by $A(G/H) = C(\lambda(G/H))$. Then each $A(G/H)$ is a cohomology theory in the sense of Cartan, and A is a G-cohomology theory over R. Note that here $\lambda_A = \lambda$, and therefore, by Theorem 1.4, $H^*(A_G(K)) \cong H_G^*(K; \lambda)$.

Example 1.8. Let C be a cohomology theory over $\mathbb{Q}$ in the sence of Cartan, and $\lambda:\mathbf{O}_G \to \mathbb{Q}$**-mod** be the rational coefficient system defined as follows:

$$\lambda(G/H) = \mathrm{Hom}(\mathbb{Q}(G/H),\mathbb{Q}),$$

where $\mathbb{Q}(G/H)$ is the vector space over $\mathbb{Q}$ generated by the set G/H, and

$$\lambda(\hat{g}) = \mathrm{Hom}(\mathbb{Q}(\hat{g}), id),$$

where $\hat{g}$ is a morphism in $\mathbf{O}_G$. Then $A:\mathbf{O}_G \to \mathscr{C}_{\mathbb{Q}}$ defined by

$$A(G/H) = \lambda(G/H) \otimes C$$

is a G-cohomology theory, where $\lambda(G/H)$ is considered as a simplicial differential graded algebra concentrated in dimension zero. Then, as before, $H^*(A_G(K)) \cong H^*_G(K;\lambda_A)$. Observe that here we have $\lambda_A = \lambda \otimes (Z^0 C)_0$.

The proofs of Theorem 1.4 and 1.5 appear in § 5. The method is based on a study of the homotopical algebra of the category $\mathbf{sRO}_G$**-mod**, and on a classification theorem for equivariant cohomology. These prerequisites are presented in § 2 through 4. Finally, in § 6 we show that for a suitable choice of G-cohomology, where G is finite, Theorem 4.9 of Triantafillou [12] can be recovered from Theorem 1.5.

2. G-simplicial sets and equivariant cohomology

A G-simplicial set K is a simplicial set together with an action of G on K by simplicial maps, regarding G as a constant simplicial group $\underline{G}$ with $\underline{G}_n = G$ for all $n \geqslant 0$, and all the face and degeneracy maps the identity map of G. This makes each K_n a G-set, and the face and degeneracy maps commute with the action of G.

A simplicial version of the equivariant Bredon–Illman cohomology [1, 6] for K may be described as follows. Let $\lambda \in \mathbf{RO}_G$**-mod**, and $C^n(K;\lambda)$ be the R-module of functions c defined on n-simplexes $x \in K_n$ such that $c(x) \in \lambda(G/G_x)$, where G_x is the isotropy subgroup at x. The inclusion $G_x \subseteq G_{d_i x}$ gives rise to a morphism $G/G_x \to G/G_{d_i x}$ in $\mathbf{O}_G$, and hence a homomorphism of R-modules $\lambda(G/G_{d_i x}) \to \lambda(G/G_x)$ which we shall denote by $\lambda(d_i x \to x)$. Define homomorphism $d: C^n(K;\lambda) \to C^{n+1}(K;\lambda)$ by

$$d(c)(x) = \sum_{i=0}^{n+1} (-1)^i \lambda(d_i x \to x) c(d_i x).$$

This makes $C^*(K;\lambda)$ a cochain complex. Next define an action of G on $C^n(K;\lambda)$ by

$$(gc)(x) = \lambda(\hat{g})(c(g^{-1}x)),$$

where $c \in C^n(K;\lambda)$, $x \in K_n$, and $\lambda(\hat{g}):\lambda(G/G_{g^{-1}x}) \to \lambda(G/G_x)$ is the isomorphism induced by the conjugacy relation $g^{-1}G_x g = G_{g^{-1}x}$. Let $C^n_G(K;\lambda)$ denote the submodule of G-invariant cochains $(C^n(K;\lambda))^G$. It is easily verified that $d(C^n_G(K;\lambda)) \subseteq C^{n+1}_G(K;\lambda)$. Define the equivariant cohomology of K with coefficient system λ by

$$H^n_G(K;\lambda) = H_n(C^*_G(K;\lambda)).$$

To complete the definition of the cohomology theory, let us note that a G-simplicial map $f:K \to L$ between G-simplicial sets induces a cochain map $f^\#:C^*_G(L;\lambda) \to C^*_G(K;\lambda)$ defined by $f^\#(c)(x) = \lambda(fx \to x)c(fx)$, where $\lambda(fx \to x):\lambda(G/G_{fx}) \to \lambda(G/G_x)$ is the homomorphism induced by the inclusion $G_x \subseteq G_{fx}$. Therefore we have a homomorphism

$$f^*:H^*_G(L;\lambda) \to H^*_G(K;\lambda).$$

The following theorem relates the Bredon–Illman cohomology $\hat{H}^*_G(X;\lambda)$ of a G-space X [6] with the cohomology $H^*_G(SX;\lambda)$, of the associated singular G-simplicial set SX.

Theorem 2.1. *Let X be a G-space with G discrete, and $\lambda \in$ **RO**$_G$**-mod**. Then there is an isomorphism*

$$\hat{H}^*_G(X;\lambda) \cong H^*_G(SX;\lambda),$$

which is functorial with respect to X.

Proof. Note that $\hat{H}^*_G(X;\lambda)$ is the homology of a cochain complex $\hat{S}^*_G(X;\lambda)$, where $\hat{S}^n_G(X;\lambda)$ is the R-module of functions c on equivariant singular n-simplexes $T:\Delta^n \times G/H \to X$ satisfying $c(T) \in \lambda(G/H)$, and certain compatibility condition (see [6], Ch. 1, Def. 4.3). We shall exhibit an isomorphism of cochain complexes

$$C^*_G(SX;\lambda) \cong \hat{S}^*_G(X;\lambda).$$

Let $T:\Delta^n \times G/H \to X$ be an equivariant singular n-simplex in X. Then $\sigma_T:\Delta^n \to X$ with $\sigma_T(x) = T(x, eH)$ is a singular n-simplex in X; that is, a simplex of the singular G-simplicial set SX. Moreover $H \subseteq G_{\sigma_T}$, for if $h \in H$ then $(h\sigma_T)(x) = h\sigma_T(x) = hT(x, eH) = T(x, eH) = \sigma_T(x)$. Thus we have a homomorphism $\lambda(G_{\sigma_T} \to H):\lambda(G/G_{\sigma_T}) \to \lambda(G/H)$. Now define $\alpha: C^n_G(SX;\lambda) \to \hat{S}^n_G(X;\lambda)$ by setting $\alpha(c)(T) = \lambda(G_{\sigma_T} \to H)c(\sigma_T)$. Next, define a homomorphism $\alpha':\hat{S}^n_G(X;\lambda) \to C^n_G(SX;\lambda)$ as follows. Let $\sigma:\Delta^n \to X$ be a singular n-simplex in X. Then $T_\sigma:\Delta^n \times G/G_\sigma \to X$ given by $T_\sigma(x, gG_\sigma) = g\sigma(x)$ is an equivariant singular n-simplex in X. We set $\alpha'(c)(\sigma) = c(T_\sigma)$. One sees easily that α and α' are well-defined cochain maps inverse to one another. ■

The following theorem, which we shall use in § 5, provides an alternative description of the cohomology groups $H^*_G(K;\lambda)$ (cf. Bredon [1], ch. I, § 9). Given a G-simplicial set K, define $\underline{C}_n(K) \in$ **RO**$_G$**-mod**, for each integer $n \geq 0$, in the following way

$$\underline{C}_n(K)(G/H) = C_n(K^H; R),$$

where $C_n(K^H; R)$ denotes the free R-module generated by the n-simplexes of K^H, and, for a G-map $\hat{g}:G/H \to G/H'$ induced by a subconjugacy relation $g^{-1}Hg \subseteq H'$,

$$\underline{C}_n(K)(\hat{g}) = g*,$$

where g^* is the chain map induced by the left translation $g:K^{H'} \to K^H$. Clearly this gives a chain complex $\underline{C}_*(K)$ (where the boundary $\partial:\underline{C}_n(K;R) \to \underline{C}_{n-1}(K;R)$ is coming from $\partial(G/H):C_n(K^H;R) \to C_{n-1}(K^H;R)$) in the abelian category **RO**$_G$**-mod,** and if $\lambda \in$ **RO**$_G$**-mod**, then $\mathrm{Hom}(\underline{C}_*(K), \lambda)$, which is the R-module of natural transformations $\underline{C}_*(K) \to \lambda$, becomes a cochain complex of R-modules.

Theorem 2.2. *There is an isomorphism*

$$C^*_G(K;\lambda) \cong \mathrm{Hom}(\underline{C}_*(K), \lambda)$$

of cochain complexes.

Proof. Associate with each $c \in C^n_G(K;\lambda)$ a natural transformation

$$\varphi(c):\underline{C}_n(K) \to \lambda$$

as follows. If $x \in K_n^H$, then $H \subseteq G_x$, and this induces a homomorphism $\lambda(G_x \to H)$: $\lambda(G/G_x) \to \lambda(G/H)$. Then $\varphi(c)(G/H): C_n(K^H; R) \to \lambda(G/H)$ is the homomorphism

$$\varphi(c)(G/H)(x) = \lambda(G_x \to H)c(x).$$

This gives $\varphi: C_G^n(K; \lambda) \to \mathrm{Hom}(\underline{C}_n(K), \lambda)$. Next, define its inverse $\varphi': \mathrm{Hom}(\underline{C}_n(K), \lambda) \to C_G^n(K; \lambda)$ as follows. If $T: \underline{C}_n(K) \to \lambda$ is a natural transformation and $x \in K_n$, then

$$\varphi'(T)(x) = T(G/G_x)(x).$$

It can be checked without difficulty that φ and φ' are well-defined cochain maps inverse to each other. ∎

3. Closed model structure of $\mathbf{sRO}_G$-mod

First note that the category $\mathbf{sRO}_G$**-mod** may be identified with the category of contravariant functors $\mathbf{O}_G \to$ **sR-mod.** Then, by a result of Dwyer and Kan [3,4], $\mathbf{sRO}_G$**-mod** is a closed model category in the sense of Quillen [8] with the following structures: A morphism $f: T \to S$ is a fibration (resp. weak equivalence) if for every $G/H \in \mathbf{O}_G$ the simplicial map $f(G/H): T(G/H) \to S(G/H)$ is a fibration (resp. weak equivalence), and f is a cofibration if it satisfies LLP (left lifting property) with respect to trivial fibrations. Also it follows easily that

Lemma 3.1. *Every object in* $\mathbf{sRO}_G$**-mod** *is fibrant as well as cofibrant.* ∎

Note that an object $T \in \mathbf{sRO}_G$**-mod** is fibrant (resp. cofibrant) if the morphism $T \to \underline{0}$ (resp. $\underline{0} \to T$) is a fibration (resp. cofibration), where $\underline{0}$ is the initial object in $\mathbf{sRO}_G$**-mod.**

We shall now briefly discuss the homotopy theory in $\mathbf{sRO}_G$**-mod.** There are two notions of homotopy in $\mathbf{sRO}_G$**-mod**: (i) the left homotopy coming from its closed model structure, and (ii) the abstract homotopy coming from combinatorial considerations as described in [7], § 5. We shall show that the two notions are essentially the same.

First let us look at the abstract notion of homotopy in $\mathbf{sRO}_G$**-mod.** Let $\underline{RI}: \mathbf{O}_G \to$ **sR-mod** be the contravariant functor defined by $\underline{RI}(G/H) = RI$ and $\underline{RI}(\hat{g}) = id$, where RI is the free simplicial R-module generated by $I = \Delta[1]$. Note that if $\{e_0, e_1, \ldots, e_{n+1}\}$ is the basis of the R-module $(RI)_n$, where $e_k = (0, 0, \ldots, 0, 1, \ldots, 1)$ (with $n - k + 1$ zeros and k ones) $\in \Delta[1]_n$, then

$$(RI)_n = Re_0 \oplus \cdots \oplus Re_{n+1}.$$

If $T: \mathbf{O}_G \to$ **sR-mod** is another contravariant functor, define a contravariant functor $T \otimes \underline{RI}: \mathbf{O}_G \to$ **sR-mod** by $T \otimes \underline{RI}(G/H) = T(G/H) \otimes RI$. Also, define natural transformations $i_0, i_1: T \to T \otimes \underline{RI}$ by

$$i_0(G/H)(x) = x \otimes (e_0, 0, \ldots, 0) \quad \text{and} \quad i_1(G/H)(x) = x \otimes (0, \ldots, e_{n+1}).$$

We then obtain the following

Lemma 3.2. *Two simplicial maps* $f, g: T \to S$ *in* $\mathbf{sRO}_G$**-mod** *are homotopic (in the abstract sense) if and only if there exists a simplicial map*

$$F: T \otimes \underline{RI} \to S$$

with $F \circ i_0 = f$ *and* $F \circ i_1 = g$. ∎

Lemma 3.3. *Every homotopy equivalence in* **sRO$_G$-mod** *is a weak equivalence.* ■

Now we turn to the notion of left homotopy in a closed model category $\mathscr{C}$. Let $A \vee_{\varnothing} A$ be the push out of the diagram $A \leftarrow \varnothing \rightarrow A$ in $\mathscr{C}$ and

$$\nabla_A : \mathrm{A} \vee A \rightarrow A$$

be the corresponding folding map. Recall from [8], ch. I, § 1 than in $\mathscr{C}$, a cylinder of an object A is an object IA together with morphisms $i_0, i_1 : A \rightarrow IA$ and $p : IA \rightarrow A$ such that $i_0 + i_1 : A \vee_{\varnothing} A \rightarrow IA$ is a cofibration, p is a weak equivalence, and $p \circ (i_0 + i_1) = \nabla_A$. Two morphisms $f_0, f_1 : A \rightarrow B$ in $\mathscr{C}$ are called left homotopic ($f_0 \sim_c f_1$) if there is a morphism $H : IA \rightarrow B$ such that $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$. Quillen proved that if A is cofibrant then i_0 and i_1 are trivial cofibrations, and the left homotopy relation $\sim_c$ is an equivalence relation.

In the closed model category **sRO$_G$-mod** every object is cofibrant (Lemma 3.1), the initial object ϕ is just $\underline{0}$, and, for an object T, $T \vee_{\varnothing} T$ is simply $T \oplus T$ with the folding map $\nabla_T : T \oplus T \rightarrow T$ given by $\nabla_T(G/H)(x, x') = x + x'$. We define $IT = T \otimes \underline{RI}$, the natural transformations $i_0, i_1 : T \rightarrow IT$ as in Lemma 3.2, and $p : IT \rightarrow T$ by $p(G/H)(x \otimes u) = x$. Then the natural transformation $i_0 + i_1 : T \oplus T \rightarrow IT$ is given by

$$(i_0 + i_1)(G/H)(x, x') = i_0(G/H)(x) + i_1(G/H)(x'),$$

and we have $p \circ (i_0 + i_1) = \nabla_T$. Also, p is a homotopy equivalence in the abstract sense with homotopy inverse i_0. Therefore, by Lemma 3.3, p is a weak equivalence in **sRO$_G$-mod**.

Again $i_0 + i_1$ is a cofibration. To see this, consider a LLP of $i_0 + i_1$ with respect to a trivial fibration $q : U \rightarrow V$ in **sRO$_G$-mod**.

$$\begin{tikzcd} T \oplus T \arrow[r, "\alpha"] \arrow[d, "i_0 + i_1"'] & U \arrow[d, "q"] \\ T \otimes \underline{RI} \arrow[r, "\beta"] & V \end{tikzcd}$$

We may identify $T \otimes \underline{RI}$ with $T \oplus T \oplus S$, where $S = \operatorname{coker}(i_0 + i_1)$, by means of a splitting of the exact sequence

$$\underline{0} \rightarrow T \oplus T \rightarrow T \otimes \underline{RI} \rightarrow S \rightarrow \underline{0},$$

(note that $i_0 + i_1$ is injective). Also, the LLP of the confibration $\underline{0} \rightarrow S$ with respect to the trivial fibration $q : U \rightarrow V$ has a solution $\gamma : S \rightarrow U$ such that $q \circ \gamma = \beta$. Then a solution to the LLP of $i_0 + i_1$ is given by $\alpha + \gamma : T \oplus T \oplus S \rightarrow U$. Thus we have proved the following lemma.

Lemma 3.4. *In the category* **sRO$_G$-mod,** $T \otimes \underline{RI}$ *is a cylinder object for* T. ■

Theorem 3.5. *Two morphisms $f, g : T \rightarrow S$ in* **sRO$_G$-mod** *are left homotopic if and only if they are homotopic in the abstract sense. Consequently, the homotopy between morphisms in the abstract sense is an equivalence relation.* ■

4. A classification theorem in $\mathbf{sRO}_G$-mod

In this section we shall show that if $\lambda \in \mathbf{RO}_G$**-mod**, then for any G-simplicial set K there is a bijection between the cohomology group $H^n_G(K;\lambda)$ and the set $[\Phi K, K(\lambda, n)]$ of homotopy classes of morphisms in $\mathbf{sRO}_G$**-mod,** where ΦK is as in Definition 1.3, and $K(\lambda, n)$ is what we call an $\mathbf{O}_G$-Eilenberg–MacLane complex of the type (λ, n). This a contravariant functor $T:\mathbf{O}_G \to \mathbf{sR}$**-mod** such that

(1) $T(G/H)$ is an Eilenberg–MacLane complex $K(\lambda(G/H), n)$,
(2) $T(\hat{g}): T(G/H) \to T(G/H')$ is the unique simplicial homomorphism induced by the linear map $\lambda(\hat{g}): \lambda(G/H) \to \lambda(G/H')$, $g^{-1}H'g \subseteq H$.

The first condition means that

$$\underline{\pi}_n \circ T = \lambda, \quad \text{and} \quad \underline{\pi}_j \circ T = \underline{0} \quad \text{if} \quad j \neq n,$$

where $\underline{\pi}_n(K)$ is defined by $\underline{\pi}_n(K)(G/H) = \pi_n(K^H)$. It may be noted that each $K(\lambda(G/H), n)$ is minimal by definition, and that the Eilenberg–MacLane G-simplicial set may be obtained from $K(\lambda, n)$ by applying a functorial bar construction (see [5]), and as in ([5], Cor., p. 280) we have the following

Theorem 4.1. *Any two $\mathbf{O}_G$-Eilenberg–MacLane complexes of type (λ, n) are naturally isomorphic.*

Proof. Suppose that $T, S:\mathbf{O}_G \to \mathbf{sR}$**-mod** are two $\mathbf{O}_G$-Eilenberg–MacLane complexes of type (λ, n). Define a natural transformation $\eta: T \to S$ by setting $\eta(G/H): T(G/H) \to S(G/H)$ to be the unique simplicial homomorphism induced by $id: \lambda(G/H) \to \lambda(G/H)$. Since there is a bijection

$$[T(G/H), S(G/H)] \leftrightarrow \mathrm{Hom}(\lambda(G/H), \lambda(G/H)),$$

$\varphi(G/H)$ is a homotopy equivalence, and hence an isomorphism, because a homotopy equivalence between two minimal Kan complexes is an isomorphism. Now the following diagram commutes.

$$\begin{tikzcd}
T(G/H) \arrow[r, "\varphi(G/H)"] \arrow[d, "T(\hat{g})"'] & S(G/H) \arrow[d, "S(\hat{g})"] \\
T(G/H') \arrow[r, "\varphi(G/H')"] & S(G/H')
\end{tikzcd}$$

This is so because $S(\hat{g}) \circ \varphi(G/H)$ and $\varphi(G/H') \circ T(\hat{g})$ are the unique simplicial maps induced by $\lambda(\hat{g}) \circ id_{\lambda(G/H)}$ and $id_{\lambda(G/H')} \circ \lambda(\hat{g})$ respectively. This proves the theorem. ■

From now on we shall consider only normalized chain and cochain complexes (see May [7]). Fix $n \geq 0$, and define contravariant functors

$$L(\lambda, n+1), K(\lambda, n): \mathbf{O}_G \to \mathbf{sR}\text{-}\mathbf{mod}$$

by setting

$$L(\lambda, n+1)(G/H)([q]) = C^n(\Delta[q]; \lambda(G/H)),$$

$$K(\lambda, n)(G/H)([q]) = Z^n(\Delta[q]; \lambda(G/H)),$$

where $[q]$ denotes the ordered set $\{0<1\ldots<q\}$. Then $K(\lambda,n)$ is an $\mathbf{O}_G$-Eilenberg–MacLane complex of the type (λ,n). Define a map

$$\Lambda:\mathrm{Hom}(\Phi K, L(\lambda,n+1))\to\mathrm{Hom}(\underline{C}_n(K),\lambda)$$

as follows. Let $f:\Phi K\to L(\lambda,n+1)$ be a natural transformation. Then $f(G/H):RK^H\to L(\lambda,n+1)(G/H)$ is a simplicial map. If $x\in K_n^H$, then $f(G/H)(x)\in L(\lambda,n+1)(G/H)[n]=C^n(\Delta[n];\lambda(G/H))$ is a cochain. Since $C^n(\Delta[n];\lambda(G/H))$ is the R-module of all linear transformations from the R-module $R\Delta_n$ with basis Δ_n to $\lambda(G/H)$, we may identify it with $\lambda(G/H)$. Then define $\Lambda f:\underline{C}_n(K)\to\lambda$ by

$$(\Lambda f)(G/H)(x)=(f(G/H)(x))(\Delta_n).$$

It is straightforward to check that Λf is natural with respect to morphisms in $\mathbf{O}_G$.

Next define

$$\Lambda':\mathrm{Hom}(\underline{C}_n(K),\lambda)\to\mathrm{Hom}(\Phi K,L(\lambda,n+1))$$

as follows. Let $T:\underline{C}_n(K)\to\lambda$ be a natural transformation. Then it is sufficient to define simplicial map

$$\Lambda'(T)(G/H):RK^H\to L(\lambda,n+1)(G/H).$$

Let $x\in RK_q^H$. This induces a simplicial map $\bar{x}:R\Delta[q]\to RK^H$ with $\bar{x}(\Delta_q)=x$. Then

$$\bar{x}^*:C^n(RK^H;\lambda(G/H))\to C^n(R\Delta[q];\lambda(G/H))$$

is a cochain map. Observe that $C^n(R\Delta[q];\lambda(G/H))=L(\lambda,n+1)(G/H)([q])$. We then set

$$(\Lambda'T)(G/H)(x)=\bar{x}^*(T(G/H)).$$

It can be verified that $\Lambda'T$ is a natural transformation, and that Λ and Λ' are inverse to each other. We have thus proved the following

Lemma 4.2. *The map Λ is an isomorphism between the functors* $\mathrm{Hom}(\Phi K,L(\lambda,n+1))$ *and* $\mathrm{Hom}(\underline{C}_n(K),\lambda)$ *with inverse Λ'.* ∎

Let $\varphi':\mathrm{Hom}(\underline{C}_n(K),\lambda)\to C_G^n(K;\lambda)$ denote the isomorphism of Theorem 2.2, with inverse φ. Denote the composition

$$\varphi'\circ\Lambda:\mathrm{Hom}(\Phi K,L(\lambda,n+1))\to C_G^n(K;\lambda)$$

by Γ. Then we obtain

Lemma 4.3. *The map Γ is an isomorphism between the cocycles $Z_G^n(K;\lambda)$ and* $\mathrm{Hom}(\Phi K,K(\lambda,n))$ *with inverse $\Gamma'=\Lambda'\circ\varphi$.*

Proof. Let $f\in Z_G^n(K;\lambda)$. We need to show that $(\varphi f)(G/H)(x)\in K(\lambda(G/H),n)$, for all $x\in K_q^H$ and $H\subseteq G$, that is, $(\varphi f)(G/H)(x)\in Z^n(\Delta[q];\lambda(G/H))$. But this is true, because

$$\begin{aligned}\delta(\Gamma'T(G/H)(x))&=\delta(\Lambda'(\varphi T)(G/H)(x))\\&=\delta(\theta_H(id\times\hat{e})^*\tilde{x}^*(\widehat{\varphi T}))\\&=\delta(\theta_H(id\times\hat{e})^*\tilde{x}^*T)=0.\end{aligned}$$

Conversely, suppose that $(\varphi f)(G/H)(y) \in Z^n(\Delta[q]; \lambda(G/H))$ for all $y \in K_q^H$. Then, if $x \in K_{n+1}$ is non-degenerate, we find after a simple computation that $\delta(\Gamma f)(x) = 0$. This proves the lemma. ■

Theorem 4.4. *Let $f_0, f_1 \in \mathrm{Hom}(\Phi K, K(\lambda, n))$. Then $f_0 \sim f_1$ if and only if Γf_0 and Γf_1 are cohomologous.*

Proof. Let $F: \Phi K \otimes RI \to K(\lambda, n)$ be a homotopy between f_0 and f_1 (see § 3). Therefore for each $H \subseteq G$, the simplicial maps $f_0(G/H)$ and $f_1(G/H)$ are homotopic by the homotopy $F(G/H)$. Define an element u_H in the n-cochain group $C^n(L(\lambda, n+1)(G/H); \lambda(G/H))$ by setting $u_H(c) = c(\Delta_n)$. This gives a homomorphism

$$f(G/G_x)^*: C^n(L(\lambda, n+1)(G/G_x); \lambda(G/G_x)) \to C^n(RK^{G_x}; \lambda(G/G_x))$$

such that

$$\begin{aligned} f(G/G_x)^*(u_{G_x})(x) &= u_{G_x}(f(G/G_x)(x)) \\ &= f(G/G_x)(x)(\Delta_n) \\ &= \Gamma f(x). \end{aligned}$$

Now since the simplicial maps $f_0(G/G_x)$ and $f_1(G/G_x)$ are homotopic, $f_0(G/G_x)^* = f_1(G/G_x)^*$. Consequently $\Gamma f_0 = \Gamma f_1$.

Conversely, suppose that f_0, $f_1: \Phi K \to K(\lambda, n)$ are such that Γf_0 and Γf_1 are cohomologous, that is, $\Gamma f_0 = \Gamma f_1 + dh$, where $h \in C_G^{m-1}(RK; \lambda)$. It suffices to find a $\gamma \in Z_G^n(RK \otimes RI; \lambda)$ such that $i_0^*(\gamma) = \Gamma f_0$ and $i_1^*(\gamma) = \Gamma f_1$ where $i_0, i_1: RK \to RK \otimes RI$ are the inclusions as in Lemma 3.2. Then the natural transformation

$$\Gamma'(\gamma): \Phi K \otimes \underline{RI} \to K(\lambda, n)$$

will be a homotopy from f_0 to f_1. To get such a γ, write $\gamma_0 = p^*(\Gamma f_0) \in Z_G^n(RK \otimes RI; \lambda)$, where p is projection $RK \otimes RI \to RK$. Then

$$i_0^*(\gamma_0) = i_1^*(\gamma_0) = \Gamma f_0.$$

Further, regarding $h \in C_G^{n-1}(RK; \lambda)$ as a cochain defined on $i_1(RK)$, we may choose a cochain $\beta \in C_G^{n-1}(RK \otimes RI; \lambda)$ which extends h and vanishes on $i_0(RK)$. Thus $i_0^*(\beta) = 0$ and $i_1^* \beta = h$. Now take $\gamma = \gamma_0 - d\beta$. This completes the proof. ■

We have in effect proved the following theorem.

Theorem 4.5. (*Classification*) *For any G-simplicial set K, there is a bijection*

$$[\Phi K, K(\lambda, n)] \leftrightarrow H_G^n(K; \lambda).$$

5. Proofs of Theorems 1.4 and 1.5

We begin by proving a lemma. Recall from § 1 that given a G-cohomology theory $A: \mathbf{O}_G \to \mathscr{C}_R$, each $A(G/H)$ is a cohomology theory in the sense of Cartan, and we have a $\lambda_A \in \mathbf{RO}_G$**-mod** defined by $\lambda_A(G/H) = (Z^0(A(G/H)))_0$, where $Z^0(A(G/H))$ is the kernel of the homomorphism $\delta_H: A^0(G/H) \to A^1(G/H)$. We also have contravariant functors

$$A^n, Z^n A: \mathbf{O}_G \to \mathbf{sR\text{-}mod},$$

where $A^n(G/H)$ and $Z^n A(G/H)$ are simplicial R-modules with the set of p-simplexes as $A(G/H)^n_p$ and $\operatorname{Ker}\{\delta: A(G/H)^n_p \to A(G/H)^{n+1}_p\}$ respectively.

Lemma 5.1. *If $A:\mathbf{O}_G \to \mathscr{C}_R$ is a G-cohomology theory over R, then each $A^n:\mathbf{O}_G \to$ **sR-mod** is contractible as an object of* **sRO$_G$-mod**.

Proof. For every $H \subseteq G$, and $n \geqslant 0$, the exact sequence $A^0(G/H) \xrightarrow{\delta} A^1(G/H) \to \ldots$ gives rise to a short exact sequence

$$0 \to Z^n A(G/H) \to A^n(G/H) \to Z^{n+1} A(G/H) \to 0.$$

This amounts to saying that $A^n(G/H) \to Z^{n+1} A(G/H)$ is a principal fibration with fibre $Z^n A(G/H)$ in the category of simplicial sets, and hence a principal twisted cartesian product (PTCP) of type (W) with group complex $Z^n A(G/H)$ in the sense of May [7], where $A^n(G/H)$ is contractible. This PTCP of type (W) is naturally isomorphic to the universal PTCP of type (W), $W(Z^n A(G/H)) \to \bar{W}(Z^n A(G/H))$, constructed by means of $\bar{W}$- and W-constructions on $Z^n A(G/H)$. It can be checked easily that $W(Z^n A(G/H))$ is contractible and the contraction can be chosen so as to be natural with respect to the morphisms in $\mathbf{O}_G$. Consequently, the contraction of $A^n(G/H)$ is also natural. The resulting contractions are all the necessary equipments one needs for the construction of a contraction of $A^n:\mathbf{O}_G \to$ **sR-mod.** ∎

Proof of Theorem 1.4. It is required to prove that there is an isomorphism

$$H^*_G(K;\lambda_A) \cong H^*(A_G(K)),$$

where $A_G(K)$ is the differential graded algebra $\operatorname{Hom}(\Phi K, A^n)$ of Definition 1.3. Note that $Z^n A:\mathbf{O}_G \to$ **sR-mod** is an $\mathbf{O}_G$-Eilenberg–MacLane complex of the type (λ_A, n), and that for $n > 0$ we have a short exact sequence

$$0 \to Z^n A \to A^n \to Z^{n+1} A \to 0,$$

in the category **sRO$_G$-mod.** We may therefore identify $Z^n A_G(K) = \operatorname{Ker}(A^n_G(K) \to A^{n+1}_G(K))$ with $\operatorname{Hom}(\Phi K, Z^n A)$, which is the R-module of morphisms from ΦK to $Z^n A$. There is an obvious map

$$\operatorname{Hom}(\Phi K, Z^n A) \to [\Phi K, Z^n A] \cong H^n_G(K;\lambda_A),$$

where the isomorphism is as given in Theorem 4.5. We shall show that if $f \in \operatorname{Hom}(\Phi K, Z^n A)$ is homotopic to constant, then it factors through $p: A^{n-1} \to Z^n A$. Consider a commutative diagram:

$$\begin{tikzcd} \Phi K \arrow[r] \arrow[d, "i_0"'] & A^{n-1} \arrow[d, "p"] \\ \Phi K \otimes \underline{RI} \arrow[r, "F"] & Z^n A \end{tikzcd}$$

where the horizontal map on the top is the constant map, F is a homotopy between f and the constant map, the vertical map i_0 on the left is a trivial cofibration. Since p is surjective, it is a fibration. Consequently, since **sR-mod** is a closed model category, the above LLP of i_0 with respect to p has a solution

$$\tilde{F}: \Phi K \otimes \underline{RI} \longrightarrow A^{n-1}$$

such that $p \circ \tilde{F} | i_1(\Phi K) = f$. This proves the theorem when $n > 0$.

For $n=0$, it is easy to see that, since $Z^0 A_G(K) = \mathrm{Hom}(\Phi K, Z^0 A)$, two morphisms f, $g \in \mathrm{Hom}(\Phi K, Z^0 A)$ are homotopic if and only if they are equal. This completes the proof of the theorem. ■

Proof of Theorem 1.5. Given $\lambda \in \mathbf{RO}_G$-**mod**, consider the contravariant functor $A: \mathbf{O}_G \to \mathscr{C}_R$ defined by

$$A(G/H) = C^*(\Delta[\ ]; \lambda(G/H)),$$

where $C^p(\Delta[q]; \lambda(G/H))$ denotes the ordinary singular cochain group. Then $A(G/H)$ is a cohomology theory in the sense of Cartan and A is a G-cohomology theory with $\lambda = \lambda_A$. Set $\hat{A}_G(X) = A_G(SX)$. The proof now follows from Theorems 2.2 and 1.4. ■

6. Equivariant commutative cochain problem

We conclude the paper with the observation that for a suitable choice of G-cohomology theory Theorem 1.4 leads to Theorem 4.9 of Triantafillou [12]. Let G be a finite group, $\lambda: \mathbf{O}_G \to \mathbb{Q}$-**mod** a (contravariant) rational coefficient system, and $H_G^*(X; \lambda)$ the Bredon cohomology of a G-complex X. Let $\underline{\mathscr{E}}_X: \mathbf{O}_G \to \mathbf{CDGA}/\mathbb{Q}$ be the covariant functor, where $\underline{\mathscr{E}}_X(G/H) = \mathscr{E}_{X^H}$ which is the de Rham algebra over $\mathbb{Q}$ of rational polynomial forms on X^H, and, for a morphism $\hat{g}: G/H \to G/H'$ in $\mathbf{O}_G$, $g^{-1}Hg \subset H'$, $\underline{\mathscr{E}}_X(\hat{g}) = g^*: \mathscr{E}_{X^H} \to \mathscr{E}_{X^{H'}}$ which is induced by the left translation $g: X^{H'} \to X^H$. Let λ^* and $\underline{\mathscr{E}}_X^*$ denote respectively the functors dual to λ and $\underline{\mathscr{E}}_X$. Then according to Triantafillou [12], Theorem 4.9,

$$H^*(\mathrm{Hom}(\lambda^*, \underline{\mathscr{E}}_x)) \cong H_G^*(X; \lambda).$$

On the other hand, Theorem 1.5 gives

$$H^*(\mathrm{Hom}(\Phi X, \lambda \otimes_{\mathbb{Q}} \underline{\mathscr{E}}_X^*)) \cong H_G^*(X; \lambda).$$

It is not difficult to see that the cochain complexes $\mathrm{Hom}(\lambda^*, \underline{\mathscr{E}}_X)$ and $\mathrm{Hom}(\Phi X, \lambda \otimes_{\mathbb{Q}} \underline{\mathscr{E}}_X^*)$ are isomorphic. Note that we have $R(\lambda \otimes_{\mathbb{Q}} \underline{\mathscr{E}}_X^*) = \lambda$.

References

[1] Bredon G E, Equivariant cohomology theories. *Springer Lecture Notes in Math.* **34** (1967)
[2] Cartan H, Théories cohomologiques, *Invent. Math.* **35** (1976) 261–271
[3] Dwyer W G and Kan D M, Function complexes of diagrams of simplicial sets, *Indag. Math.* **45** (1983) 139–147
[4] Dwyer W G and Kan D M, A classification theorem for diagrams of simplicial sets, *Topology* **23** (1984) 139–155
[5] Elmendorf A D, Systems of fixed point sets, *Trans. Am. Math. Soc.* **227** (1983) 275–284
[6] Illman S, Equivariant singular homology and cohomology, *Mem. Am. Math. Soc.* **156** (1975)
[7] May J P, *Simplicial objects in algebraic topology*, (New York: Van Nostrand) (1967)
[8] Quillen D G, Homotopical algebra, *Springer Lecture Notes in Math.* **43** (1967)
[9] Quillen D G, Rational homotopy theory, *Ann. Math.* **90** (1969) 205–295
[10] Sullivan D, Infinitesimal computations in topology, *Publ. Math. IHES* **47** (1978) 269–331
[11] Swan R G, Thom's theory of differential forms on a simplicial set. *Topology* **14** (1975) 271–273
[12] Triantafillou G, Equivariant minimal models, *Trans. Am. Math. Soc.* **274** (1982) 509–532

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 163–167.

On non-fragmentability of Banach spaces

A K MIRMOSTAFAEE

Department of Mathematics, Damghan Faculty of Sciences,
P.O. 36715/364, Damghan, Iran

MS received 12 September 1997; revised 24 March 1998

Abstract. In this paper, we use the game characterization of Kenderov and Moors [11] to construct an example of a non-fragmentable Banach space. More precisely, we will show that if X is the tree-complete Banach algebra of Haydon and Zizler [3], (X/c_0, weak) is not fragmentable by any metric. In particular, this shows that X/c_0 cannot be equivalently renormed to be rotund.

Keywords. Fragmentability of Banach spaces; topological games.

1. Introduction

Following Jayne and Rogers [9], we call a topological space (X,τ) *fragmentable* if there exists a metric $d(.,.)$ on X such that for every nonempty set $A \subset X$ and $\varepsilon > 0$ there exists a nonempty subset $B \subset A$ which is τ-relatively open in A and d-diam$(B) := \sup\{d(x', x''): x', x'' \in B\} < \varepsilon$. In such a case it is said that the metric $d(.,.)$ fragments X. Evidently, every metric space is fragmented by its metric. However, the class of fragmentable spaces is much larger than the class of metrizable spaces.

In [11] the following topological game was used to characterize fragmentability of the space X. Two players Σ and Ω alternatively select subsets of X. Σ starts the game by selecting an arbitrary non-empty subset A_1 of X. Then Ω chooses some non-empty subset B_1 of A_1 which is relatively open in A_1. In general, if the selection $B_n \neq \emptyset$ of the player Ω is already specified, the player Σ makes the next move by selecting an arbitrary nonempty set $A_{n+1} \subset B_n$. In return, Ω selects a nonempty relatively open subset B_{n+1} of A_{n+1}. Continuing this alternative selection of sets in X a sequence $A_1 \supset B_1 \supset A_2 \supset \cdots A_n \supset B_n \supset \cdots$ is generated which we call a *play* and denote by $p = (A_i, B_i)_i$. The player Ω is said to have won a play $p = (A_i, B_i)_i$ if the set $\bigcap_{i \geqslant 1} A_i = \bigcap_{i \geqslant 1} B_i$ contains at most one point. Otherwise the player Σ is said to have won this play. If $p_n = (A_1, B_1, \ldots, A_n, B_n, \ldots, A_n)$ are the first n moves of some play (of the game), then we call p_n a *partial play* of the game. A *strategy* ω for the player Ω is a mapping that assigns to each partial play p_n some nonempty relatively open subset $\omega(A_1, B_1, \ldots, A_n)$ of A_n. Given the strategy ω, we call a play $p = (A_i, B_i)_i$ an ω-*play* if $B_i = \omega(A_1, B_1, \ldots, A_i)$ for every $i \geqslant 1$. i.e. p is an ω-play if Ω makes his/her selections by means of (or according to) the strategy ω. The strategy ω is called a *winning strategy* for Ω if every ω-play is won by the player Ω. If the space X is fragmentable by a metric $d(.,.)$, then Ω has an obvious winning strategy ω. Indeed, to each partial play $A_1 \supset B_1 \supset \cdots \supset A_k$ this strategy puts into correspondence some nonempty subset $B_n \subset A_n$ which is relatively open in A_n and has d-diameter less than $1/n$. Clearly, the set $\bigcap_{i \geqslant 1} A_i = \bigcap_{i \geqslant 1} B_i$ has at most one point because it has d-diameter 0. It turns out that the existence of winning strategy for the player Ω characterizes the fragmentability of Ω.

Theorem 1.1. ([11], Theorem 1.1) *The topological space X is fragmentable if, and only if, the player Ω has a winning strategy for the above game.*

The notion of fragmentability and its variants have many interesting connections with other topics such as locally uniformly convex and Kadec renormings of Banach spaces, generic differentiability of convex functions, the Radon–Nikodym property, etc. (see [13–15, 12, 4–8]). Ribarska [14] showed that if a Banach space admits an equivalent strictly convex norm, then it is fragmentable (with respect to the weak topology). Thus non-fragmentable Banach spaces cannot be renormed to be a strictly convex Banach space. Until this paper, there were only a few known examples of non-fragmentable Banach spaces (see e.g. [7], Example 3; [11], Theorem 2.3; and [2], Theorem 3.6). In [10], the authors have constructed a larger class of non-fragmentable spaces. Bourgain [1] showed that l^∞/c_0 does not admit equivalent strictly convex norm. In this paper, we exhibit an extra example of a non-fragmentable Banach space. More precisely, we show that $(X/c_0$, weak) is not fragmented by any metric, where X is the tree-complete Banach algebra of Haydon and Zizler. It follows then that X/c_0 cannot be renormed (by an equivalent norm) to be strictly convex.

2. Results

Let $T=\bigcup_{n=1}^{\infty}\{0,1\}^n$, the set of all finite strings of 0's and 1's. The length $|t|$ of a string t is n if $t\in\{0,1\}^n$. The tree order is defined by $s\prec t$ if $|s|<|t|$ and $t(m)=s(m)$ for $m\leqslant|s|$. Each $t\in T$ has exactly two immediate successors, i.e. $t0$ and $t1$.

A lattice L is said to be *tree-complete*, if whenever $\{f_t\}_{t\in T}$ is a bounded disjoint family in X, there exists $b\in\{0,1\}^N$, such that $\Sigma_{n\in N}f_{b|n}$ is in L. Haydon and Zizler [3] constructed a closed linear subspace of l^∞ (which is a tree-complete sublattice of l^∞) such that it contains c_0 but does not contain any subspace isomorphic to l^∞. Notice that in this space X every infinite subset M of N, has an infinite subset $M_0\subset M$ such that $\chi_{M_0}\in X$ [3].

Lemma 2.1. *Let X be the tree-complete lattice of Haydon and Zizler, $\mu > X^*$, $\varepsilon>0$ and M be an infinite subset of N, then there exists an infinite subset M_0 of M such that $\chi_{M_0}\in X$ and*

$$\|f\|\leqslant 2 \textit{ and } \mathrm{supp}(f)\subset M_0 \textit{ imply that } |\mu(f)|<\varepsilon.$$

Proof. Let $2\|\mu\|<n\varepsilon$. If the lemma were not true, for every infinite subset M_0 of M, where $\chi_{M_0}\in X$, there exists an element $f\in X$, such that $\|f\|\leqslant 2$, $\mathrm{supp}(f)\subset M_0$ but $\mu(f)\geqslant\varepsilon$. Let $M_1, M_2,\ldots, M_n$ be disjoint infinite subsets of M. Suppose that $f_1, f_2,\ldots,f_n$ are elements of X such that for each $0\leqslant i\leqslant n$, $\|f_i\|\leqslant 2$, $\mathrm{supp}(f_i)\subset M_i$ and $\mu(f_i)>\varepsilon$. Then, if we put $f=\Sigma_{i=1}^n f_i$, since $f_1,f_2,\ldots,f_n$ have disjoint support we have $\|f\|\leqslant 2$ but

$$n\varepsilon>2\|\mu\|\geqslant\mu(f)=\sum_{i=1}^{n}\mu(f_i)\geqslant n\varepsilon,$$

which is a contradiction.

Theorem 2.1. *Let X be the tree-complete lattice of Haydon and Zizler, then $(X/c_0$, weak) is not fragmented by any metric.*

Proof. By Theorem 1.1, it is enough to show that the player Ω does not have a winning strategy. More precisely, we have to show that there exists a play in which player Σ wins. By induction on $|t|$, for every $t \in T$, we will construct a partial play $p_t = (A_{t|1}, B_{t|1}, A_{t|2}, \dots, A_t)$, then we will show that there exists some $b \in \{0,1\}^N$ such that the player Σ will win the play $p_b = (A_{b|1}, B_{b|1}, \dots, A_{b|n}, \dots)$.

Let $\pi: X \to X/c_0$ be the quotient map. Choose two infinite disjoint subsets N'_0 and N'_1 of N such that $\chi_{N'_i} \in X$, let $f_i = \chi_{N \setminus N'_i}$, $i = 0, 1$. Choose points $m_i \in N'_i$ and put $N_i = N'_i \setminus \{m_i\}$, then the sets

$$A_1 = \{\pi(f): 0 \leqslant f \leqslant 1,\ f(x) = f_i(x), \quad x \notin N_i\}, \quad (i = 0, 1),$$

would be the first movements of player Σ and we have partial plays $p_0 = (A_0)$ and $p_1 = (A_1)$. Thus, the first step of our induction is done.

Suppose that for every t with $|t| \leqslant n$, the partial play $p_t = (A_{t|1}, B_{t|1}, A_{t|2}, \dots, A_t)$ has already been constructed. Let B_t be the response of player Ω to A_t, let $f'_t \in B_t$, then there are linear functionals $\mu_1^t, \dots, \mu_{k_t}^t$ in X^* and $\varepsilon_t > 0$, such that

$$\{\pi(f) \in A_t : |\mu_i^t(f - f'_t)| < \varepsilon_t,\ 1 \leqslant i \leqslant K_t\} \subset B_t.$$

Applying the Lemma 2.1, once again, we can find $N'_t \subset N_t$ such that $\chi_{N'_t} \in X$, $\|f\| \leqslant 2$ and $\operatorname{supp}(f) \subset N'_t$ imply $|\mu_i^t(f)| < \varepsilon_t$ for all $1 \leqslant i \leqslant K_t$. Choose two infinite disjoint subset of N'_t, say N'_{t0} and N'_{t1} such that $\chi_{N'_{ti}} \in X$ for $i = 0, 1$, let $f_{ti} = f'_t \cdot \chi_{N \setminus N'_{ti}}$. Choose $m_{ti} \in N'_{ti}$ and put $N_{ti} = N'_{ti} \setminus \{m_{ti}\}$, then the sets

$$A_{ti} = \{\pi(f) \in A_t : f(x) = f_{ti}(x),\ x \notin N_{ti}\}, \quad (i = 0, 1)$$

are subsets of B_t and $p_{ti} = (A_{ti|1}, B_{ti|1}, \dots, A_t, B_t, A_{ti})$ for $i = 0, 1$ are partial plays. Thus, by induction on $|t|$, we proved that we can construct partial plays $p_t = (A_{t|1}, B_{t|1}, A_{t|2}, \dots, A_t)$ for each $t \in T$, such that the following conditions hold:

(i) The player Σ will choose sets of the form

$$A_t = \{\pi(f): 0 \leqslant f \leqslant 1,\ f(x) = f_t(x), \quad x \notin N_t\}$$

for each $t \in T$, such that

(ii) $N_t \subset N_s$ when $s \prec t$,

(iii) $N_t \cap N_s = \emptyset$, when s and t are not comparable,

(iv) $\operatorname{supp}(f_t) \subset N \setminus N_t$,

(v) whenever $s \prec t$, $f_t(x) = f_s(x)$ if $x \notin N_s$,

(vi) for every $t \in T$, there is $m_t \in N_t \setminus N_{ti}$, $i = 0, 1$, such that $f_t(m_t) = 0$.

We now define a bounded, disjoint family $\{g_t\}_{t \in T}$ in X by putting

$$g_{t|1} = f_{t|1}$$

$$g_{ti} = f_{ti} - f_t = \chi_{N_t \setminus N_{ti}} \cdot f_{ti}, \quad (i = 0, 1).$$

By tree-completeness, there exists some $b \in \{0,1\}^N$ such that

$$f_b(x) = \sum_{n \in N} g_{b|n}(x)$$

is in X. Note that $\pi(f_b) \in A_{b|n}$ and $f_b(m_{b|n}) = 0$ for every n. By (ii), (iii) and (vi), $\{m_t\}_{t \in T} \cap (N \setminus N_t)$ is finite for each t. Let $h = \chi_M \in X$ for the infinite set $M = \{m_t\}_{t \in T}$. Define $g = f_b + h$ and for every $n \in N$ define the function k_n by

$$k_n(x) = \begin{cases} f_b(x) & \text{if} \quad x \notin N_{b|n} \\ g(x) & \text{if} \quad x \in N_{b|n} \end{cases}$$

which belongs to $A_{b|n}$, since $M \cap (N_{b|n})^c$ for each n is finite $k_n(x) - g(x) \in c_0$ i.e., $\pi(k_n) = \pi(g) \in A_{b|n}$ for eahc n, in other words $\pi(g) \in \cap A_{b|n}$. On the other hand $g - f_b = h$ is a characteristic function of the infinite set M, so that $\pi(g) \neq \pi(f_b)$ and the intersection $\cap A_{b|n}$ contains more than one point. This shows that the player Ω does not have a winning strategy.

COROLLARY 2.1

X/c_0 has no equivalent strictly convex renorming.

Proof. Ribarska [14] proved that if in a topological space (Y, τ) there exists a strictly convex norm which is lower semi-continuous with respect to τ then (Y, τ) is a fragmentable space. Since (X/c_0, weak) is not fragmented by any metric, and the norm is lower semi-continuous wiith respect to the weak topology, this space cannot admit any equivalent strictly convex norm.

Acknowledgement

The author wishes to thank Prof. P S Kenderov for helpful discussions. Thanks are also due to the referee for his careful reading of the manuscript and useful suggestions.

References

[1] Bourgain J, l^∞/c_0 has no equivalent strictly convex norm *Proc. Am. Math. Soc.* **78(2)** (1980) 285–295
[2] Giles J R, Kenderov P S, Moors W B and Sciffer S, Generic differentiability of convex functions defined on the dual of a Banach space, *Pacific J. Math.* **172** (1996) 413–431
[3] Haydon R G and Zizler V, *A new Banach space with no locally uniformly rotund renorming, Canad. Bull. Math.* **32** (1989) 122–128
[4] Jayne J E, Namioka I and Rogers C A, Topological properties of Banach spaces, *Proc. London Math. Soc.* **66** (1993) 651–672
[5] Jayne J E, Namioka I and Rogers C A, σ-Fragmentable Banach spaces I, *Mathematika* **39** (1992) 161–188
[6] Jayne J E, Namioka I and Rogers C A, σ-Fragmentable Banach spaces II, *Mathematika* **39** (1992) 197–215
[7] Jayne J E, Namioka I and Rogers C A, Fragmentability and σ-fragmentability, *Fund. Math.* **143** (1993) 207–220
[8] Jayne J E, Namioka I and Rogers C A, Norm fragmented weak* compact sets, *Collect. Math.* **41** (1990) 133–163
[9] Jayne J E and Rogers C A, Borel selectors for upper semi-continuous set-valued maps, *Acta Math.* **56** (1985) 41–79
[10] Kenderov P S and Mirmostafaee A K, *Nonfragmentability of Banach spaces*, accepted for publication in *C. R. l' Acad. Bulgare des Sci.* (1997)
[11] Kenderov P S and Moors W B, Game characterization of fragmentability of topological spaces, Mathematics and Education in Mathematics, *Proceedings of the 25th Spring Conference of the Union of Bulgarian Mathematicians, April 1996, Kazanlak, Bulgaria* (1996) pp. 8–18
[12] Namioka I, Radon-Nikodym compact spaces and fragmentability, *Mathematika* **34** (1987) 258–281

[13] Ribarska N K, Internal characterization of fragmentable spaces, *Mathematika* **34** (1987) 243–257
[14] Ribarska N K, A note on fragmentability of some topological spaces, *C. R. l'Acad. Bulgare des Sci.* **43(7)** (1990) 13–15
[15] Ribarska N K, The dual of a Gateaux smooth Banach space is weak* fragmentable, *Proc. Am. Math. Soc.* **114(4)** (1992) 1003–1008

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 169–177.

Existence of weak and strong solutions of an integrodifferential equation in Banach spaces

M KANAKARAJ

Department of Mathematics, Kumaraguru College of Technology, Coimbatore 641 006, India

MS received 22 January 1997

Abstract. A class of evolution integrodifferential equation has been studied over the analytic semigroup of operators in a Banach space. Further the existence of weak and strong solutions in Banach space have been proved and extended to a maximum interval of existence.

Keywords. Existence of solutions; semilinear evolution equation; contraction principle.

1. Introduction

Rankin [2] discussed the evolution equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au(t) = F(u(t)), \quad u(0) = \phi \quad t \geqslant 0$$

and the related integral equation

$$u(t) = T(t)\phi + \int_0^t T(t-s)F(u(s))\mathrm{d}s, \quad t \geqslant 0,$$

where $T(t)$ is the semigroup of operators generated by $-A$. The present work is closely related to the paper [2]. We consider the following evolution integrodifferential equation in a Banach space X:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au(t) = f(u(t)) + \int_0^t k(s, u(s))\mathrm{d}s, \quad t \geqslant 0 \tag{1}$$

$$u(0) = v, \tag{2}$$

where A is a closed linear operator which is densely defined and $-A$ generates an analytic semigroup $T(t){:}t \geqslant 0$. The non-linear operators f and k are defined on $D(A^\alpha)$ for some $0 < \alpha < 1$ and $v \in D(A)$. We consider non-linear operators f and k which can be written in the form $A^\alpha G_1$ and $A^\alpha G_2$, where G_1 and G_2 maps a subspace Y of X into X. The existence of solution to (1) is closely associated with the existence of solution to the integral equation

$$u(t) = T(t)v + \int_0^t T(t-s)f(u(s))\mathrm{d}s + \int_0^t T(t-s)\left(\int_0^s k(\tau, u(\tau))\mathrm{d}\tau\right)\mathrm{d}s,$$

$$t \geqslant 0. \tag{3}$$

In the last chapter we discuss how the theory can be applied to partial differential equations.

2. Assumptions

(i) X and Y are Banach spaces with $Y \subset X$.
(ii) $-A$ generates an analytic semigroup of linear operators $T(t)$, $t \geqslant 0$ on X.
(iii) $A^{-\alpha}$ for $0 < \alpha < 1$ is defined by the integral

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\alpha} s^{\alpha-1} T(s) \mathrm{d}s,$$

where $\Gamma(\alpha)$ denotes the gamma function. Then $A^{\alpha} = (A^{-\alpha})^{-1}$ exists as a closed and densely defined linear operator $A^{-\alpha}: X \to D(A^{\alpha})$ is a bounded linear operator, where $D(A^{\alpha})$ is the domain of A^{α}. Denote X_{α} as the Banach space created by norming $D(A^{\alpha})$ with the graph norm.
(iv) There exists $M \geqslant 1$ and $\omega \in R$ such that for all $\alpha \geqslant 0$, $\|T(t)\| \leqslant M e^{\omega t}$ and $\|A^{\alpha} T(t)\|_X \leqslant C_{\alpha} t^{-\alpha} e^{\omega t}$ for some $C_{\alpha} > 0$. We shall assume $X_{\alpha} \subseteq Y \subseteq X$ so that $T(t): X \to Y$ for all $t \geqslant 0$ is a bounded linear operator.
(v) $T(t)y$ is continuous in t in the norm of Y for each $y \in Y$.
(vi) $A^{\beta} T(t) = T(t/2) A^{\beta} T(t/2): X \to Y$, for $t > 0$ and $\|A^{\beta} T(t)\| \in L^1(0, r)$ for $\beta \in [\alpha, \alpha + d]$ for some $d > 0$ and every $r > 0$.
(vii) Define the map $f: X_{\alpha} \to X$ as follows: There exists a locally Lipschitz $G_1: Y \to X$ and $G_1: X_{\alpha} \to X_{\alpha}$ satisfying for each $v \in X_{\alpha}$, $f(v) = A^{\alpha} G_1(v)$ and $\|G_1(u) - G_1(v)\| \leqslant \lambda_1 \|u - v\|$, $0 < \lambda_1 < 1$.
(viii) The function k maps $X_{\alpha} \to X$, satisfying: there exists $G_2: Y \to X$ such that G_2 is locally Lipschitz, $G_2: X_{\alpha} \to X_{\alpha}$ and for each $x \in X_{\alpha}$, $A^{\alpha} G_2(x, u(x)) = k(x, u(x))$ and $\|G_2(x, u(x)) - G_2(x, v(x))\| \leqslant \lambda_2 \|u(x) - v(x)\|$ for $u, v \in X_{\alpha}$ and $0 < \lambda_2 < 1$.
(ix) For $0 < \lambda_1 < 1$, $0 < \lambda_2 < 1$, $0 \leqslant s < t < L$, $(\lambda_1 + \lambda_2 L) \int_0^t \|A^{\alpha} T(t-s)\| \mathrm{d}s < N$, $0 < N < 1$.

3. Main results

Theorem 1. *If the assumptions* (i) *through* (viii) *hold, then for each* $v \in Y$ *there exists a* $L > 0$ *and a unique continuous function* $u: [0, L) \to Y$ *such that*

$$u(t) = T(t)v + \int_0^t T(t-s) f(u(s)) \mathrm{d}s + \int_0^t T(t-s) \left(\int_0^s k(\tau, u(\tau)) \mathrm{d}\tau \right) \mathrm{d}s, \quad t \geqslant 0.$$

Proof. Define the mapping P on E by

$$(Pu)(t) = T(t)v + \int_0^t A^{\alpha} T(t-s) G_1(u(s)) \mathrm{d}s + \int_0^t A^{\alpha} T(t-s) \left(\int_0^s G_2(\tau, u(\tau)) \mathrm{d}\tau \right) \mathrm{d}s,$$

where $E = \{u: [0, L] \to Y/u(t)$ is continuous, $u(0) = v$ and $\sup \|u(t) - v\| \leqslant R\}$, $0 \leqslant t < L$

$$\left\| \int_0^t A^{\alpha} T(t-s) G_1(u(s)) \mathrm{d}s + \int_0^t A^{\alpha} T(t-s) \left\{ \int_0^s G_2(\tau, u(\tau) \mathrm{d}\tau \right\} \mathrm{d}s \right\|$$

$$\leqslant \int_0^t \|A^{\alpha} T(t-s)\| \{ \|G_1(V)\| + \lambda_1 R \} \mathrm{d}s$$

$$+\int_0^t \|A^\alpha T(t-s)\|\left(\int_0^s \|G_2(\tau,u(\tau))\|\,\mathrm{d}\tau\right)\mathrm{d}s$$

$$\leqslant \int_0^t \|A^\alpha T(t-s)\|\,(\|G_1(V)\|+\lambda_1 R+L\{\|G_2(\tau,v)\|+\lambda_2 R\})\,\mathrm{d}s,$$

$$0\leqslant t\leqslant s\leqslant L$$

for each $u\in E$ the R.H.S. of the above inequality can be made less than R choosing $L>0$ small enough, this proves $L(E)\subseteq E$. Let $u_1(t), u_2(t)\in E$

$$\begin{aligned}\|(Pu_1)(t)-(Pu_2)(t)\| &= \Bigg\|\int_0^s T(t-s)\{G_1(u_1(s))-G_1(u_2(s))\}\,\mathrm{d}s\\
&\quad+\int_0^t A^\alpha T(t-s)\left\{\int_0^s G_2(\tau,u_1(\tau))-G_2(\tau,u_2(\tau))\,\mathrm{d}\tau\right\}\mathrm{d}s\Bigg\|\\
&\leqslant \int_0^t \|A^\alpha T(t-s)\|\lambda_1 \sup_{0\leqslant t\leqslant L}\|u_1(t)-u_2(t)\|\,\mathrm{d}s\\
&\quad+\int_0^t \|A^\alpha T(t-s)\|\int_0^s \lambda_2 \sup_{0\leqslant t\leqslant L}\|u_1(t)-u_2(t)\|\,\mathrm{d}\tau\\
&\leqslant \sup_{0\leqslant t<L}\|u_1(t)-u_2(t)\|\,N.\end{aligned}$$

Therefore P is a contraction. By contraction mapping theorem P has a unique fixed point $u\in E$. This fixed point satisfies the integral equation (3).

Lemma 1. *If the conditions* (i) *to* (ix) *hold and if* $\|A^\alpha T(t)\|\leqslant ct^{-\zeta}$ *for* $t\in(0,L], 0<\zeta<1$ *for some* $c>0$ *then the solution of* (3) *is locally Hölder continuous in* Y *norm in* $(0,L]$.

Proof.

$$u(t)=T(t)v+\int_0^t A^\alpha T(t-s)G_1(u(s))\,\mathrm{d}s+\int_0^t A^\alpha T(t-s)\int_0^s G_2(\tau,u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.$$

For all $v>0$ and $t,t+h\in[v,L]$, we have

$$\begin{aligned}\|u(t+h)-u(t)\| &\leqslant \|T(t)(T(h)-I)v\|+\Bigg\|\int_0^t A^\alpha T(h)T(t-s)G_1(u(s))\,\mathrm{d}s\\
&-\int_0^t A^\alpha T(t-s)G_1(u(s))\,\mathrm{d}s+\int_t^{t+h} A^\alpha T(h)T(t-s)G_1(u(s))\,\mathrm{d}s\\
&+\int_0^t A^\alpha T(h)T(t-s)\left(\int_0^s G_2(\tau,u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s\\
&+\int_t^{t+h} A^\alpha T(h)T(t-s)\left(\int_0^s G_2(\tau,u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s\\
&-\int_0^t A^\alpha T(t-s)\left(\int_0^s G_2(\tau,u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s\Bigg\|\end{aligned}$$

$$\leqslant \| T(t)(T(h)-I)v\| + \int_0^t \| A^\alpha(T(h)-I)T(t-s)G_1(u(s))\| \mathrm{d}s$$
$$+ \int_t^{t+h} \| A^\alpha T(t+h-s)G_1(u(s))\| \mathrm{d}s$$
$$+ \int_0^t \left\| A^\alpha(T(h)-I)T(t-s)\int_0^s G_2(\tau,u(\tau))\mathrm{d}\tau \right\| \mathrm{d}s$$
$$+ \int_t^{t+h} \left\| A^\alpha T(t+h-s)\int_0^s G_2(\tau,u(\tau))\mathrm{d}\tau \right\| \mathrm{d}s$$
$$\leqslant \| A^\alpha T(t)(T(h)-I)A^{-\alpha}v\| + \int_0^t \| A^{\alpha+\varepsilon}(T(h)-I)T(t-s)A^{-\varepsilon}G_1(u(s))\| \mathrm{d}s$$
$$+ \int_t^{t+h} \| A^\alpha T(t+h-s)G_1(u(s))\| \mathrm{d}s$$
$$+ \int_0^t \left\| A^{\alpha+\varepsilon} T(t-s)\int_0^s (T(h)-I)A^{-\varepsilon}G_2(\tau,u(\tau))\mathrm{d}\tau \right\| \mathrm{d}s$$
$$+ \int_t^{t+h} \left\| A^\alpha T(t+h-s)\int_0^s G_2(\tau,u(\tau))\mathrm{d}\tau \right\| \mathrm{d}s$$
$$\leqslant \frac{c}{v^\zeta}h^\alpha \| v\| + ch^\varepsilon\{\| G_1(v)\| + \lambda_1 R + L(\| G_2(\tau,v)\| + \lambda_2 R)\} \int_0^t \| A^{\alpha+\varepsilon}T(t-s)\| \mathrm{d}s$$
$$+ ch^{1-\zeta}\{\| G_1(v)\| + \lambda_1 R + L(\| G_2(\tau,v)\| + \lambda_2 R)\}.$$

Thus, $\| u(t+h)-u(t)\| \leqslant c(v)h^\sigma$, where $0<\varepsilon<d$ and $\sigma = \min\{\alpha, 1-\zeta, \varepsilon\}$, $0<\sigma<1$. This proves the result of Lemma 1. Assuming the locally Lipschitz continuity of f and k, we can show that $u(t)$ is a solution of (1) if the solution $u(t)$ of (3) is in X_α and is Hölder continuous in X_α norm.

Lemma 2. *Assume the condition of lemma 1 hold then the solution of the equation* (3) *is in* $X_{1-\alpha}$ *for* $t\in(0,L]$.

Proof.

$$u(t) = T(t-v)u(v) + \int_v^t A^\alpha T(t-s)G_1(u(s))\mathrm{d}s$$
$$+ \int_v^t A^\alpha T(t-s)\left(\int_0^s G_2(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s$$
$$= T(t-v)u(v) + \int_v^t A^\alpha T(t-s)G_1(u(t))\mathrm{d}s$$
$$+ \int_v^t A^\alpha T(t-s)\left(\int_0^s G_2(s,u(s))\mathrm{d}\tau\right)\mathrm{d}s$$
$$+ \int_v^t A^\alpha T(t-s)\{G_1(u(s)) - G_1(u(t))\}\mathrm{d}s$$
$$+ \int_v^t A^\alpha T(t-s)\left\{\int_0^s \{G_2(\tau,u(\tau)) - G_2(s,u(s))\}\mathrm{d}\tau\right\}\mathrm{d}s.$$

Since $T(t-v)u(v)\in D(A)$ for all $t>v$ and

$$\int_v^t A^\alpha T(t-s)G_1(u(t))\mathrm{d}s = A^{\alpha-1}\int_v^t AT(t-s)G_1(u(t))\mathrm{d}s$$

$$= A^{\alpha-1}\{G_1(u(t)) - T(t-v)G_1(u(\tau))\}$$

$$\left\|A^{1-\alpha}\int_v^t A^\alpha T(t-s)\left\{\int_0^s G_2(s,u(s)\mathrm{d}\tau\right\}\mathrm{d}s\right\|$$

$$\leqslant \int_v^t \|AT(t-s)\|\{\|G_2(s,v)\| + \lambda_2 R\}L\mathrm{d}s$$

$$\left\|A^{1-\alpha}\int_v^t A^\alpha T(t-s)\{G_1(u(s)) - G_1(u(t))\}\mathrm{d}s\right\|$$

$$\leqslant \int_v^t \|AT(t-s)\|\,\|G_1(u(s)) - G_1(u(t))\|\,\mathrm{d}s$$

$$\leqslant c\lambda_1\int_v^t (t-s)^{-1}\|u(s)-u(t)\|\,\mathrm{d}s$$

$$\leqslant c\lambda_1\frac{(t-v)^\sigma}{\sigma}.$$

Thus to prove $u(t)$ is in $X_{1-\alpha}$ for $t\in(0,L]$, we have to verify

$$\int_v^t A^\alpha T(t-s)\left\{\int_0^s G_2(\tau,u(\tau)) - G_2(s,u(s))\mathrm{d}\tau\right\}\mathrm{d}s \text{ is in } X_{1-\alpha}$$

$$\left\|A^{1-\alpha}\int_v^t A^\alpha T(t-s)\left\{\int_0^s G_2(\tau,u(\tau)) - G_2(s,u(s)\mathrm{d}\tau\right\}\mathrm{d}s\right\|$$

$$\leqslant c\lambda_2\int_v^t (t-s)^{-1}\int_0^s \|u(\tau)-u(s)\|\,\mathrm{d}\tau\,\mathrm{d}s$$

$$\leqslant c\lambda_2\int_v^t (t-s)^\sigma \mathrm{d}s$$

$$\leqslant c\lambda_2(t-v)^{\sigma+1}/(\sigma+1).$$

Lemma 3. *Assume that the condition of the previous lemma hold and that $X_{1-\alpha}\subseteq X_\alpha$, the imbedding being continuous, then the solution $u(t)$ of equation* (3) *is a mild solution of equation* (1) *and is locally Hölder continuous in X_α.*

Proof. Lemma 2 and the assumption $X_{1-\alpha}\subseteq X_\alpha$ implies that $u(t)\in X_\alpha$ for all $t>0$, $v>0$. Thus for $t>v/2$, $t+h$, $t\in[v,L]$,

$$u(t) = T(t-v/2)u(v/2) + \int_{v/2}^t A^\alpha T(t-s)G_1(u(s))\mathrm{d}s$$

$$+\int_{v/2}^t A^\alpha T(t-s)\int_0^s G_2(\tau,u(\tau))\mathrm{d}\tau\mathrm{d}s.$$

Since $u(t)$ is continuous in $X_{1-\alpha}$, so $u(t)$ is continuous in X_α and

$$\begin{aligned}\|u(t+h)-u(t)\| &\leqslant \|(T(h)-I)A^\alpha T(t-v/2)u(v/2)\| \\ &\quad + \int_{v/2}^{t} \|A^\alpha\{T(h)-I\}T(t-s)f(u(s))\|\,\mathrm{d}s \\ &\quad + \left\|\int_{v/2}^{t} \|A^\alpha\{T(h)-I\}T(t-s)\int_0^s k(\tau,u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s\right\| \\ &\quad + \int_t^{t+h} \|A^\alpha T(t+h-s)f(u(s))\|\,\mathrm{d}s \\ &\quad + \int_t^{t+h} \left\|A^\alpha T(t+h-s)\int_0^s k(\tau,u(\tau))\,\mathrm{d}\tau\right\|\mathrm{d}s, \qquad (4)\end{aligned}$$

$$\begin{aligned}\|\{T(h)-I\}A^\alpha T(t-v/2)u(v/2)\| &\leqslant \|\{T(h)-I\}AT(t-v/2)A^{\alpha-1}u(v/2)\| \\ &\leqslant \frac{c}{(t-v/2)}\|(T(h)-I)A^{\alpha-1}u(v/2)\| \\ &\leqslant ch^\alpha\|u(v/2)\|, \qquad (4a)\end{aligned}$$

$$\begin{aligned}&\left\|\int_{v/2}^{t} A^\alpha T(t-s)\{T(h)-I\}f(u(s))\,\mathrm{d}s\right\| \\ &\quad\leqslant \int_{v/2}^{t} \|A^{\alpha+\varepsilon}T(t-s)\{T(h)-I\}A^{-\varepsilon}f(u(s))\|\,\mathrm{d}s \\ &\quad\leqslant \int_{v/2}^{t} \|A^{\alpha+\varepsilon}T(t-s)\|\int_0^h A^{1-\alpha}T(s)\|\,\|f(u(s))\|\,\mathrm{d}s \\ &\quad\leqslant c\int_{v/2}^{t} \|A^{\alpha+\varepsilon}T(t-s)\|\,\mathrm{d}s h^\varepsilon \sup_{v/2\leqslant t\leqslant L}\|f(u(t))\|, \qquad (4b)\end{aligned}$$

$$\begin{aligned}&\left\|\int_{v/2}^{t} A^\alpha T(t-s)\{T(h)-I\}\int_0^s k(\tau,u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s\right\| \\ &\quad\leqslant \int_{v/2}^{t} L\|A^{\alpha+\varepsilon}T(t-s)\| \sup_{v/2\leqslant\tau\leqslant L}\|k(\tau,u(\tau))\|h^\varepsilon, \qquad (4c)\end{aligned}$$

$$\left\|\int_t^{t+h} A^\alpha T(t+h-s)f(u(s))\,\mathrm{d}s\right\| \leqslant \frac{ch^{1-\alpha}}{(1-\alpha)}\sup_{v/2\leqslant t\leqslant L}\|f(u(t))\|, \qquad (4d)$$

$$\begin{aligned}&\left\|\int_t^{t+h} A^\alpha T(t+h-s)\int_0^s k(\tau,u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s\right\| \\ &\quad\leqslant \frac{cL}{(1-\alpha)}\sup_{v/2\leqslant\tau\leqslant L}\|k(\tau,u(\tau))\|h^{1-\alpha}. \qquad (4e)\end{aligned}$$

By using (4a)–(4e), equation (4) becomes,

$$\begin{aligned}\|u(t+h)-u(t)\|_{X_\alpha} &\leqslant c\|u(v/2)\|h^\alpha \\ &\quad + c\int_{v/2}^{t}\|A^{\alpha+\varepsilon}T(t-s)\|\,\mathrm{d}s \sup_{v/2\leqslant t\leqslant L}\|f(u(t))\|h^\varepsilon\end{aligned}$$

$$+cL\int_{v/2}^{t} \|A^{\alpha+\varepsilon}T(t-s)\| \sup_{v/2\leqslant\tau\leqslant L} \|k(\tau,u(\tau))\| h^{\varepsilon} ds$$

$$+\frac{c}{(1-\alpha)} \sup \|f(u(t))\| h^{1-\alpha} + \frac{cL}{(1-\alpha)} \sup_{v/2\leqslant\tau\leqslant L} \|k(\tau,u(\tau))\| h^{1-\alpha},$$

where ε is chosen so that $\alpha+\varepsilon<1$. Thus there exist a $c>0$ and a $0<\mu<1$ such that $\|u(t+h)-u(t)\|_{X_\alpha}\leqslant ch^{\mu}$, for $t+h, t\in[v,L]$. Hence $u(t)$ is locally Hölder continuous in X_α norm.

Theorem 2. *Assume the conditions* (i) *to* (viii), *the assumptions of Lemma 3, and that f and k are locally Lipschitz from X_α into X, then any solution of* (3) *is also a strong solution of equation* (1) *for $t>0$.*

Proof. Since f is locally Lipschitz and $u(t)$ is locally Hölder into X_α, the function $f(u(t))$ is locally Hölder continuous on $[v,L]$ for any $v>0$. Thus the theory of analytic semigroups of linear operators [1] gives the desired result.

Theorem 3. *Let the conditions of Lemma 1 hold then $u(t)$ may be extented to a maximum interval of existence $[0,L_{\max}]$. If $L_{\max}<\infty$ then*

$$\operatorname*{Lt}_{t\to L_{\max}} \int_0^t \|A^{\alpha}T(t-s)G_1(u(s))\| ds = \infty,$$

$$\operatorname*{Lt}_{t\to L_{\max}} \int_0^t\int_0^s \|T(t-s)k(\tau,u(\tau))\| d\tau ds = \infty \tag{5}$$

and

$$\operatorname*{Lt}_{t\to L_{\max}} \|u(t)\| = \infty. \tag{6}$$

Proof. Suppose $L_{\max}<\infty$, for each $t\in[0,L_{\max}]$, $u(t)$ satisfies the integral equation (3) claim $\operatorname*{Lt}_{t\to L_{\max}} \sup \|u(t)\|\leqslant c$ for all $t\in[0,L_{\max})$ and some $c>0$. This implies

$$\|u(t)\| \leqslant \|T(t)v\| + \left(\int_0^{L_{\max}} \|A^{\alpha}T(t-s)\| ds\right)\left\{\sup_{0\leqslant t\leqslant L} \|G_1(t)\|\right\}$$
$$+L\left(\int_0^{L_{\max}} \|A^{\alpha}T(t-s)\| ds\right)\left\{\sup_{0\leqslant s\leqslant L_{\max}} \|G_2(s,u(s)\|\right\}.$$

Now for $0<y<t<L_{\max}$, we have

$$\|u(t)-u(y)\| \leqslant \|A^{\alpha}T(y)\| \, \|T((t-y)-I)A^{-\alpha}v\|$$
$$+\int_0^y \|A^{\alpha+\varepsilon}T(y-s)\| \, \|(T(t-y)-I)A^{-\varepsilon}G_1(u(s))\|$$
$$+\int_0^y\int_0^s \|A^{\alpha+\varepsilon}T(y-s)\| \, \|(T(t-y)-I)A^{-\varepsilon}G_2(\tau,u(\tau))\| d\tau ds$$
$$+\int_y^t \|A^{\alpha}T(t-s)G_1(u(s))\| ds + \int_y^t\int_0^s \|A^{\alpha}T(t-s)G_2(\tau,u(\tau))\|$$

$$\leqslant \frac{c}{y^{\zeta}}(t-y)^{\alpha}\|v\| + c(t-y)^{\varepsilon}\left\{\int_0^y \|A^{\alpha+\varepsilon}T(y-s)\|\,\mathrm{d}s\right\}\left\{\sup_{0\leqslant t\leqslant L_{\max}}\|G_1(u(t))\|\right\}$$

$$+c(t-y)^{\varepsilon}L\left\{\int_0^y \|A^{\alpha+\varepsilon}T(y-s)\|\,\mathrm{d}s\right\}\sup_{0\leqslant s\leqslant L_{\max}}\|G_2(s,u(s))\|$$

$$+\int_y^t \|A^{\alpha}T(s)\|\,\mathrm{d}s\left\{\sup_{0\leqslant t\leqslant L_{\max}}\|G_1(u(t))\|\right\}$$

$$+L\left\{\int_y^t \|A^{\alpha}T(s)\|\,\mathrm{d}s\right\}\sup_{0\leqslant s\leqslant L_{\max}}\|G_2(s,u(s))\|.$$

Thus we have $\|u(t)-u(y)\| = 0$ as $t, y\to L_{\max}$ contradicting the maximality of $L_{\max}$.

If $\operatorname{Lt}_{t\to L_{\max}} \|u(t)\| \neq \infty$ then there exist numbers $r>0$ and $d>0$ with d arbitrarily large and sequences $y_n\to L_{\max}$, $t_n\to L_{\max}$ as $n\to\infty$ and such that $y_n<t_n<L_{\max}$, $\|u(y_n)\|=r$, $\|u(t_n)\|=r+d$ and $\|u(t)\|\leqslant r+d$ for $t\in[y_n, t_n]$. We have

$$u(t_n) = T(t_n)u(y_n) + \int_{y_n}^{t_n} A^{\alpha}T(t_n-s)G_1(s)\mathrm{d}s$$

$$+\int_{y_n}^{t_n} A^{\alpha}T(t_n-s)\left\{\int_0^s G_2(\tau,u(\tau))\mathrm{d}\tau\right\}\mathrm{d}s,$$

$$\|u(t_n)-u(y_n)\| \leqslant \|\{T(t_n-y_n)-I\}u(y_n)\| + \left\|\int_{y_n}^{t_n} A^{\alpha}T(t_n-s)G_1(u(s))\mathrm{d}s\right.$$

$$\left.+\left\|\int_{y_n}^{t_n} A^{\alpha}T(t_n-s)\left\{\int_0^s G_2(\tau,u(\tau))\mathrm{d}\tau\right\}\mathrm{d}s\right\|,\qquad (7)$$

since left side of (7) is bounded below by $d>0$ and right side of (7) tends to zero as t_n, $y_n\to L_{\max}$. This contradiction gives the result and (5) follows from (6).

4. Application

Let us consider the following heat conduction equation

$$\frac{\partial u}{\partial t} - \Delta u(t) = \int_0^t \sum_{i=1}^n D_i g_i(s,x,u(s,x),\nabla u(s,x))\mathrm{d}s + f(x,u(t,x)\nabla u(t,x)),$$

$$u(t,x) = 0 \qquad \text{for} \quad (t,x)\in(0,\alpha)\times\partial\Omega,$$

$$u(0,x) = u_0(x) \quad \text{for} \quad x\in\Omega. \qquad (8)$$

Let $\Omega = R^n$ be a bounded domain with smooth boundary $\partial\Omega$. Let $D=I$ be the identity operator and let $D_i = \partial/\partial_{x_i}$, $\Delta = \Sigma_{i=0}^n D_i^2$, $\nabla = (D_1, D_2,\ldots, D_n)$ denote the gradient operator.

Assumptions

(I) For each $i=0,1,2,\ldots,n$, g_i: $[0,\infty)\times\Omega\times R\times R^n\to R$ is continuous and for every $t_0>0$, $r_0>0$ there exists $M_0>0$ such that if $\|u\|\leqslant r_0$, $\|v\|\leqslant r_0$ then

$$\|g_i(t,x,u,\zeta) - g_i(t,x,v,\eta)\| \leqslant M_0\{\|u-v\| + \|\zeta-\eta\|\} \quad \text{for all}$$

$$0\leqslant t\leqslant t_0, x\in\Omega, \zeta,\eta\in R^n, i=1,2,\ldots,n.$$

(II) The function $f:\Omega\times R\times R^n\to R$ is continuous and for every $t_0>0$, $r_0>0$ there exists $L_0>0$ such that if $\|u\|\leqslant r_0$, $\|v\|\leqslant r_0$ then

$$\|f(x,u,\zeta)-f(x,v,\eta)\|\leqslant L_0\{\|u-v\|+\|\zeta-\eta\|\}\quad\text{for}\quad x\in\Omega,\zeta,\eta\in R^n.$$

Let $n/2<p<\alpha$ and $X=L^p(\Omega)$ with the usual norm $\|u\|_p=\{\int_\Omega|u|^p\mathrm{d}x\}^{1/p}$, $W^{m,p}(\Omega)$ the Sobolev space of all functions on Ω whose distributional derivatives up through order m are in $L^p(\Omega)$ with norm given by

$$\|u\|_{m,p}=\left\{\sum_{\alpha\leqslant m}\|\partial^\alpha u/\partial x^\alpha\|_p\right\}^{1/p}.$$

Let $A=-\Delta$ denote the negative Laplacian in $L^p(\Omega)$, $D(A)=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. The operator $-A$ generates an analytic semigroup $\{T(t):t\geqslant 0\}$ in $L^p(\Omega)$ and fractional power $A^{1/2}$ is well defined. Let $X_{1/2}=D(A^{1/2})=W_0^{1,p}(\Omega)$. By factorization theorem there are bounded linear operators B_i on X such that $D_i=A^{1/2}B_i$ on $X_{1/2}$ for all $0\leqslant i\leqslant n$.

Define operators

$$q_i(t,y)=B_ig_i(t,*,y,\nabla y)\quad\text{and}\quad q(t,y)=\sum_{i=0}^{n}q_i(t,y). \tag{9}$$

By Sobolev imbedding theorem we have $W_0^{1,p}(\Omega)\to c(\bar{\Omega})$ with imbedding constant $L_1>0$, if $y\in W_0^{1,p}(\Omega)$ and if $\|y\|_{1,p}\leqslant r_0^*$ then $\|y(x)\|\leqslant L_1r_0^*$ for all $x\in\Omega$ so by assumption (I) with $r_0=L_1r_0^*$ we have given $t_0>0$ there exist $M_0>0$ such that if $y,z\in W_0^{1,p}(\Omega)$ with $\|y\|_{1,p}\leqslant r_0^*$, $\|z\|_{1,p}\leqslant r_0^*$ then

$$\|g_i(t,x,y(x),\nabla y(x))-g_i(t,x,z(x),\nabla z(x))\|\leqslant M_0\{\|y-z\|+\|\nabla y-\nabla z\|\}$$

for all $x\in\Omega$, $0\leqslant t\leqslant t_0$, $i=1,2,\ldots,n$. Thus there is a constant $M>0$ such that $\|q_i(t,y)-q_i(t,z)\|_p\leqslant M\|y-z\|_{1,p}$, q_i is a continuous mapping from $[0,\alpha)\times X_{1/2}$ to X, the operator $q(t,y)$ defined by (9) satisfies the assumption (ix). We define operator $k(s,y)=\Sigma_{i=0}^n D_ig_i(t,x,y,\nabla y)$ by (9) and the factorization $k(s,y)=A^{1/2}q(t,y)$. Let $f(u(t))$ defined on $X_{1/2}$ by $f(u(t))=f(x,u,\nabla u)$, a.e. $x\in\Omega$ and it is clear that f is a continuous mapping from $X_{1/2}$ to X and satisfies the assumption (vii). Hence for each $u_0\in W_0^{1,p}(\Omega)$, the heat conduction equation (8) can be written as evolution integrodifferential equation (1) in the Banach spaces $L^p(\Omega)$, $n/2<p<\alpha$.

Acknowledgement

The author is greatly indebted to Dr K Balachandran, Bharathiar University, Coimbatore, for suggesting the problem and providing useful guidelines.

References

[1] Pazy A, *Semigroup of linear operators and applications to partial differential equations* (New York: Springer-Verlag) (1983)

[2] Samuel M Rankin III, Semilinear evolution equations in Banach spaces with application to parabolic partial differential equations, *Trans. Am. Math. Soc.* **336** (1993) 523–535

Maximum and minimum solutions for nonlinear parabolic problems with discontinuities

DIMITRIOS A KANDILAKIS and
NIKOLAOS S PAPAGEORGIOU*
Department of Mathematics, University of the Aegean, 83200 Karlovassi Samos, Greece
*Address for correspondence: Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece

MS received 27 November 1997

Abstract. In this paper we examine nonlinear parabolic problems with a discontinuous right hand side. Assuming the existence of an upper solution φ and a lower solution ψ such that $\psi \leqslant \varphi$, we establish the existence of a maximum and a minimum solution in the order interval $[\psi, \varphi]$. Our approach does not consider the multivalued interpretation of the problem, but a weak one side "Lipschitz" condition on the discontinuous term. By employing a fixed point theorem for nondecreasing maps, we prove the existence of extremal solutions in $[\psi, \varphi]$ for the original single valued version of the problem.

Keywords. Upper solution; lower solution; evolution triple; compact embedding; integration by parts; Sobolev space; regular cone.

1. Introduction

Let $Z \subseteq R^N$ be a bounded domain with a C^1 boundary Γ. In this paper we consider the following nonlinear parabolic problem, with discontinuous right hand side f:

$$\left\{\begin{array}{l} \dfrac{\partial x}{\partial t} - \sum\limits_{k=1}^{N} D_k a_k(t,z,Dx(t,z)) = f(t,z,x(t,z)) \text{ on } T \times Z \\ x(0,z) = x_0(z) \text{ on } Z, x|_{T\times\Gamma} = 0 \end{array}\right\} . \tag{1}$$

Assuming the existence of an upper solution φ and of a lower solution ψ, for which we have $\psi \leqslant \varphi$, we prove the existence of a maximum and of a minimum solution for the problem (1) in the order interval $[\psi, \varphi]$. Our work here extends theorem 1 of Carl and Heikkilä [5], who deal with semilinear problems. Moreover our approach is different. Instead of using transfinite induction, as was done by Carl and Heikkilä [5], we appeal directly to a theorem on the extremal fixed points of a nondecreasing map, which can be found in the book of Heikkilä and Lakshmikantham [7]. Our work here is also closely related to the recent paper of Cardinali, Fiacca and Papageorgiou [4]. In that paper, the authors examine nonlinear parabolic problems with discontinuous right hand side which can have both upward and downward jumps. By filling in only the downward jumps, they pass to a multivalued problem for which an existence theory can be developed and then prove the existence of extremal solutions in the order interval $[\psi, \varphi]$. In this paper, by assuming a one sided Lipschitz condition on the discontinuous nonlinearity, we are able to develop an existence theory for the original problem and not its multivalued interpretation. The literature on the single-valued version of the problem in limited and deals only with semilinear problems (see [5] and references therein).

2. Preliminaries

In this section we fix our notation, introduce our assumptions on the data of the problem and recall some basic definitions and facts from nonlinear analysis that we will need in the sequel.

Recall that by an 'evolution triple', we understand a triple of spaces $X \subseteq H \subseteq X^*$ such that (i) X is a separable, reflexive Banach space; (ii) H is a separable Hilbert space identified with its dual (pivot space); and (iii) the embedding of X into H is continuous and dense. Then H is embedded into X^* continuously and densely (see [9], p. 416).

In what follows let $2 \leq p < \infty$ and $(1/p) + (1/q) = 1$ (conjugate exponents). Let $W^{1,p}(Z)$ be the usual Sobolev space and $W^{1,p}(Z)^*$ its dual. Then the spaces $W^{1,p}(Z) \subseteq L^2(Z) \subseteq W^{1,p}(Z)^*$ form an evolution triple for which the embeddings are also compact (Sobolev embedding theorem). Also by $W_0^{1,p}(Z)$ we denote the subspace of $W^{1,p}(Z)$ whose elements have zero trace. We know that $W^{-1,q}(Z)$ is the dual of $W_0^{1,p}(Z)$. Then $W_0^{1,p}(Z) \subseteq L^2(Z) \subseteq W^{-1,q}(Z)$ is also an evolution triple, again with the embeddings being compact. Using these two evolution triples, we can define the following two spaces:

$$\hat{W}_{pq}(T) = \left\{ f \in L^p(T, W^{1,p}(Z)) : \frac{\partial f}{\partial t} \in L^q(T, W^{1,p}(Z)^*) \right\}$$

and

$$W_{pq}(T) = \left\{ f \in L^p(T, W_0^{1,p}(Z)) : \frac{\partial f}{\partial t} \in L^q(T, W^{-1,q}(Z)) \right\}.$$

In these definitions, the time derivative of f is understood in the sense of vector valued distributions. Both these spaces equipped with the obvious norm $\|f\|_{pq} = [\|f\|^2_{L^p(T,X)} + \|f'\|^2_{L^q(T,X^*)}]^{1/2}$ with $X = W^{1,p}(Z)$ (resp. $W_0^{1,p}(Z)$) and $X^* = W^{1,p}(Z)^*$ (resp. $X^* = W^{-1,q}(Z)$) become separable, reflexive Banach spaces. Moreover both are embedded continuously in $C(T, X)$ and compactly in $L^p(T, H)$ (see [9], proposition 23.23, p. 422 and p. 450).

Our hypotheses on the functions $a_k(t, z, \eta)$ are the following:

H(a): $a_k : T \times Z \times R^N \to R$, $k \in \{1, 2, \ldots, N\}$, are functions such that

(i) $(t, z) \to a_k(t, z, \eta)$ is measurable;

(ii) $\eta \to a_k(t, z, \eta)$ is continuous;

(iii) $|a_k(t, z, \eta)| \leq \beta_1(t, z) + c_1 \|\eta\|^{p-1}$ a.e. on $T \times Z$, with $\beta_1 \in L^q(T \times Z)$ and $c_1 > 0$;

(iv) $\sum_{k=1}^N a_k(t, z, \eta)\eta_k \geq c_2 \|\eta\|^p - \beta_2(t, z)$ a.e. on $T \times Z$, with $\beta_2 \in L^1(T \times Z)$ and $c_2 > 0$;
and

(v) $\sum_{k=1}^N (a_k(t, z, \eta) - a_k(t, z, x, \eta'))(\eta_k - \eta'_k) \geq 0$ a.e. on $T \times Z$, for all $\eta, \eta' \in R^N$.

These hypotheses allow us to define the semilinear form $a : L^p(T, W^{1,p}(Z)) - L^p(T, W^{1,p}(Z)) \to R$ by

$$a(x, y) = \int_0^b \int_Z \sum_{k=1}^N a_k(t, z, Dx) D_k y(t, z) \mathrm{d}z \mathrm{d}t.$$

Also in what follows by $((\cdot, \cdot))$ we will denote the duality brackets for the pairs $(L^q(T, W^{1,p}(Z)^*), L^p(T, W^{1,p}(Z)))$ and $(L^q(T, W^{-1,q}(Z)), L^p(T, W_0^{1,p}(Z)))$. Recall that $L^p(T, W^{1,p}(Z))^* = L^q(T, W^{1,p}(Z)^*)$ and $L^p(T, W_0^{1,p}(Z))^* = L^q(T, W^{-1,q}(Z))$.

DEFINITION 2.1

A function $\varphi \in \hat{W}_{pq}(T)$ is said to be an upper solution of (1) if

$$\left(\left(\frac{\partial \varphi}{\partial t}, u\right)\right) + a(\varphi, u) \geqslant \int_0^b \int_Z f(t, z, \varphi(t, z))u(t, z)\mathrm{d}z\mathrm{d}t$$

for all $u \in L^p(T, W_0^{1,p}(Z)) \cap L^p(T \times Z)_+$, $\varphi(0, z) \geqslant x_0(z)$ a.e. on Z and $\varphi|_{T\times\Gamma} \geqslant 0$.

Similarly $\psi \in \hat{W}_{pq}(T)$ is said to be a 'lower solution' of problem (1) if the inequalities in the definition are reversed.

We will assume the following about the upper and lower solutions:

$\mathbf{H}_0$: there exist an uper solution φ and a lower solution ψ such that $\psi \leqslant \varphi$.

Now we can introduce our hypotheses on the discontinuous nonlinearly $f(t, z, x)$:

$\mathbf{H}(f)$: $f: T \times Z \times R \to R$ is a function such that

(i) $f(.,.,\psi(.,.)), f(.,.,\varphi(.,.)) \in L^q(T \times Z)$;
(ii) there exists $M > 0$ such that for almost all $(t, z) \in T \times Z$, the function $x \to f(t, z, x) + Mx$ is nondecreasing on the interval $[\psi(t, z), \varphi(t, z)]$: and
(iii) if $x \in \hat{W}_{pq}(T) \subseteq C(T, L^2(Z))$ and $\psi(t,.) \leqslant x(t,.) \leqslant \varphi(t,.)$ for all $t \in T$ (inequality in the usual order of $L^2(Z)$), then $(t, z) \to f(t, z, x(t, z))$ is measurable.

Remark 2.1 If $f(.,.,.)$ is a jointly Borel measurable functions or more generally a Shragin function (see [1], p. 17), then hypothesis $H(f)$ (iii) is satisfied. This includes the case of a Caratheodory perturbation term. Moreover by virtue of hypothesis $H(f)$ (ii) and theorem 1.9, p. 32 of [1], hypothesis $H(f)$ (ii) is satisfied if $(t, z) \to f(t, z, x)$ is measurable and there exists a Borel measurable function $\hat{f}: T \times Z \times R \to R$ such that $f(t, z, x) = \hat{f}(t, z, x)$ for all $(t, z) \in (T \times Z) \setminus N$ and all $x \in R$, where N is a null set of $T \times Z$.

The very general conditions on the functions $a_k(t, z, \eta)$, allow us to think only in terms of 'weak' solutions.

DEFINITION 2.2

A function $x \in W_{pq}(T)$ is said to be a 'solution' of problem (1), if

$$\left(\left(\frac{\partial x}{\partial t}, u\right)\right) + a(x, u) \geqslant \int_0^b \int_Z f(t, z, x(t, z))u(t, z)\mathrm{d}z\mathrm{d}t \quad .$$

for all $u \in L^p(T, W_0^{1,p}(Z))$.

On $\hat{W}_{pq}(T)$ we can consider the partial order induced by the superspace $C(T, L^2(Z))$ (recall that $\hat{W}_{pq}(T) \subseteq C(T, L^2(Z))$. So we can define the order interval $K = [\psi, \varphi] = \{y \in \hat{W}_{pq}(T): \psi(t,.) \leqslant y(t,.) \leqslant \varphi(t,.)$ for every $t \in T$ (in $L^2(Z)$)$\}$. Our goal is to find (if they exist), the greatest (maximum) and the least (minimum) solutions of (1) in the order interval K. For this we will need the following result taken from the book of Heikkila and Lakshmikantham [7] (see theorem 1.2, p. 23).

Theorem 2.1. *If $\hat{K}$ is a subset of an ordered metric space V, $[a, b]$ is a nonempty order interval in $\hat{K}$, $G: [a, b] \to [a, b]$ is a nondecreasing map such that $\{G(x_n)\}_{n\geqslant 1}$ converges in $\hat{K}$ for every monotone sequence $\{x_n\}_{n\geqslant 1}$ in $[a, b]$, then $G(\cdot)$ has a least and a greatest fixed point in $[a, b]$.*

3. Main theorem

In this section we prove the following existence theorem for problem (1):

Theorem 3.1. *If hypotheses $H(a)$, H_0 and $H(f)$ hold and $x_0 \in L^2(Z)$, then problem* (1) *has a greatest* (*maximum*) *and a least* (*minimum*) *solution in the order interval.*

Proof. Let $y \in K$ and consider the following auxiliary parabolic problem:

$$\left\{\begin{array}{l} \dfrac{\partial x}{\partial t} - \sum_{k=1}^{N} D_k a_k(t,z,Dx) + Mx(t,z) \\ = f(t,z,y(t,z)) + My(t,z) \ \text{ on } \ T \times Z \\ x(0,z) = x_0(z) \ \text{ on } \ Z, x|_{T\times\Gamma} = 0 \end{array}\right\}. \tag{2}$$

Since $y \in K$, from hypothesis $H(f)$ we get that the function $h(t,z) = f(t,z,y(t,z)) + My(t,z)$ belongs in $L^q(T \times Z)$. Also let $L: D \subseteq L^p(T, W_0^{1,p}(Z)) \to L^q(T, W^{-1,q}(Z))$ be defined by $L(x) = \partial x/\partial t$ for $x \in D = \{x \in L^p(T, W_0^{1,p}(Z)) : \partial x/\partial t \in L^q(T, W^{-1,q}(Z))$, $x(0,z) = x_0(z)$ a.e. on $Z\}$. From Lemma III, 4.1, p. 167 of [2], we know that $L(\cdot)$ is maximal monotone. Now let $A: L^p(T, W_0^{1,p}(Z)) \to L^q(T, W^{-1,q}(Z))$ be defined by

$$((A(x), y)) = a(x,y) + M(x,y)_{L^2(T\times Z)}$$
$$= \int_0^b \int_Z \sum_{k=1}^{N} a_k(t,z,Dx) D_k y \, dz dt + M \int_0^b \int_Z x(t,z) y(t,z) dz dt.$$

Evidently hypothesis $H(A)$ ensures that $A(\cdot)$ is monotone, demicontinuous and defined everywhere. Thus $A(\cdot)$ is a maximal monotone and so is $(L+A)(\cdot)$ (see [9], theorem 32.I, p. 888). Also from hypothesis $H(a)$(iv) we have that $((A(x), x)) \geqslant \hat{c}_2 \|x\|_{1,p}^p - \hat{\beta}_2$ for some $\hat{c}_2 > 0$, $\hat{\beta}_2 > 0$ (here $\|\cdot\|_{1,p}$ denotes the norm of the Sobolov space $W_0^{1,p}(Z)$) and from the integration by parts formula for functions in $W_{pq}(Z)$, we have $((L(x), x)) = 1/2 \|x(b,.)\|_2^2 - 1/2 \|x_0(\cdot)\|_2^2$ (see [9], proposition 23.23, pp. 422–423). So we infer that $(L+A)(\cdot)$ is also coercive; i.e.

$$\lim_{\|x\|_{1,p} \to \infty} \frac{((L(x) + A(x), x))}{\|x\|_{1,p}} = +\infty.$$

But a maximal monotone, coercive operator is surjective (see for example [9], corollary 32.35, p. 887). So problem (2) has a solution $x \in W_{pq}(T)$, which is easily seen to be unique because of the monotonicity of the operator $A(\cdot)$. Thus we can define a single valued map $R: K \to W_{pq}(T)$, where for each $y \in K$, $R(y) = x \in W_{pq}(T)$ is the unique solution of (2).

Claim 1. $R(K) \subseteq K$.

To this end let $y \in K$ and $x = R(y)$. Note that $(\psi - x)^+ \in L^p(T, W^{1,p}(Z)) \cap L^p(T \times Z)_+$ (see [6], lemma 7.6, p. 145). So using this as a test function and since ψ is a lower solution, we have

$$\left(\left(\frac{\partial \psi}{\partial t}, (\psi - x)^+\right)\right) + a(\psi, (\psi - x)^+) \leqslant ((h_1, (\psi - x)^+)) \tag{3}$$

with $h_1(t,z) = f(t,z,\psi(t,z)), h_1 \in L^{q'}(T \times Z)$ (see hypothesis $H(f)$ (i)). Also since $x \in W_{pq}(T)$ is a solution of (2), we have

$$\left(\left(\frac{\partial x}{\partial t}, (\psi - x)^+\right)\right) + a(x, (\psi - x)^+) + M((x, (\psi - x)^+)) = ((h, (\psi - x)^+)), \tag{4}$$

where recall that $h(t,z)=f(t,z,y(t,z))+My(t,z)$, $h\in L^q(T\times Z)$. Multiplying (3) with -1 and then adding it to (4), we obtain

$$\left(\left(\frac{\partial x}{\partial t}-\frac{\partial \psi}{\partial t},(\psi-x)^+\right)\right)+a(x,(\psi-x)^+)-a(\psi,(\psi-x)^+)$$
$$\geqslant((h-h_1-Mx,(\psi-x)^+)). \tag{5}$$

From the integration by parts formula for functions in $\hat{W}_{pq}(T)$, we have

$$\left(\left(\frac{\partial x}{\partial t}-\frac{\partial \psi}{\partial t},(\psi-x)^+\right)\right)=-\frac{1}{2}\|(\psi(b,.)-x(b,.))^+\|_2^2\leqslant 0. \tag{6}$$

In addition from hypothesis $H(a)$ (v), it follows that

$$a(x,(\psi-x)^+)-a(\psi,(\psi-x)^+)$$
$$=\int_0^b\int_Z\sum_{k=1}^N(a_k(t,z,Dx)-a_k(t,z,D\psi))D_k(\psi-x)^+\,dz\,dt\leqslant 0 \tag{7}$$

(see [6], p. 145). Moreover we have

$$((h-h_1-Mx,(\psi-x)^+))=(h-h_1-Mx,(\psi-x)^+)_{L^q,L^p}$$
$$=\int_0^b\int_Z(f(t,z,y(t,z))+My(t,z)-f(t,z,\psi(t,z))$$
$$-Mx(t,z))(\psi-x)^+(t,z)\,dz\,dt=\int_0^b\int_{\{\psi\geqslant x\}}(f(t,z,y(t,z))$$
$$+My(t,z)-f(t,z,\psi(t,z))-Mx(t,z))(\psi-x)(t,z)\,dz\,dt$$
$$\geqslant\int_0^b\int_{\{\psi\geqslant x\}}(f(t,z,y(t,z))+My(t,z)-f(t,z,\psi(t,z))$$
$$-M\psi(t,z))(\psi-x)(t,z)\,dz\,dt. \tag{8}$$

Using (6), (7) and (8) into (5), we obtain

$$0\geqslant\int_0^b\int_Z(f(t,z,y(t,z))+My(t,z)-f(t,z,\psi(t,z))$$
$$-M\psi(t,z))(\psi-x)^+(t,z)\,dz\,dt.$$

On the other hand by virtue of hypothesis $H(f)$ (ii) and since $y\in K$, we have

$$0\leqslant\int_0^b\int_Z(f(t,z,y(t,z))+My(t,z)-f(t,z,\psi(t,z))$$
$$-M\psi(t,z))(\psi-x)^+(t,z)\,dz\,dt.$$

Therefore we deduce that

$$0=\int_0^b\int_{\{\psi\geqslant x\}}(f(t,z,y(t,z))+My(t,z)-f(t,z,\psi(t,z))$$
$$-M\psi(t,z))(\psi-x)(t,z)\,dz\,dt.$$

from which it follows at once that $\psi(t,.) \leqslant x(t,.)$ for all $t \in T$ (recall that $W_{pq}(T) \subseteq C(T,H)$). Similarly we can show that $x(t,.) \leqslant \varphi(t,.)$ for all $t \in T$. So indeed $R(K) \subseteq K$.

Claim 2. $R(\cdot)$ is nondecreasing on K.

Let $y_1, y_2 \in K$ and assume that $y_1(t,.) \leqslant y_2(t,.)$ for all $t \in T$. Let $x_1 = R(y_1)$ and $x_2 = R(y_2)$. We need to show that $x_1(t,.) \leqslant x_2(t,.)$ in $L^2(Z)$ for all $t \in T$. Using $(x_1 - x_2)^+ \in L^p(T, W_0^{1,p}(Z))$ as our test function and letting $\hat{f}(y_1)(t,z) = f(t,z,y_1(t,z))$ and $\hat{f}(y_2)(t,z) = f(t,z,y_2(t,z))$, we have

$$\left(\left(\frac{\partial x_1}{\partial t}, (x_1 - x_2)^+\right)\right) + a(x_1, (x_1 - x_2)^+) + Mx_1$$
$$= ((\hat{f}(y_1) + My_1, (x_1 - x_2)^+)) \tag{9}$$

and

$$-\left(\left(\frac{\partial x_2}{\partial t}, (x_1 - x_2)^+\right)\right) - a(x_2, (x_1 - x_2)^+) - Mx_2$$
$$= -((\hat{f}(y_2) + My_2, (x_1 - x_2)^+)). \tag{10}$$

Adding (9) and (10) above we obtain

$$\left(\left(\frac{\partial x_1}{\partial t} - \frac{\partial x_2}{\partial t}, (x_1 - x_2)^+\right)\right) + a(x_1, (x_1 - x_2)^+)$$
$$- a(x_2, (x_1 - x_2)^+) + M((x_1 - x_2, (x_1 - x_2)^+))$$
$$= ((\hat{f}(y_1) + My_1 - \hat{f}(y_2) - My_2, (x_1 - x_2)^+)). \tag{11}$$

As in the proof of claim 1, via the integration by parts formula for functions in $W_{pq}(T)$, we obtain

$$\left(\left(\frac{\partial x_1}{\partial t} - \frac{\partial x_2}{\partial t}, (x_1 - x_2)^+\right)\right) = \int_0^b \int_Z \left(\frac{\partial x_1}{\partial t} - \frac{\partial x_2}{\partial t}\right)(x_1 - x_2)^+ \, dz dt$$
$$= \frac{1}{2} \| (x_1(b,.) - x_2(b,.))^+ \|_2^2 \geqslant 0, \tag{12}$$

$$a(x_1, (x_1 - x_2)^+) - a(x_2, (x_1 - x_2)^+) \geqslant 0 \text{ and}$$
$$M((x_1 - x_2, (x_1 - x_2)^+)) \geqslant 0. \tag{13}$$

Moreover from hypothesis $H(f)$ it follows that

$$((\hat{f}(y_1) + My_1 - \hat{f}(y_2) - My_2, (x_1 - x_2)^+)) \leqslant 0. \tag{14}$$

Using (12), (13) and (14) in (11) we have

$$0 = \int_0^b \int_{\{x_1 \geqslant x_2\}} (f(t,z,y_1(t,z)) + My_1,(t,z) - f(t,z,y_2(t,z))$$
$$- My_2(t,z))(x_1(t,z) - x_2(t,z)) dz dt$$

from which we deduce that $x_1(t,.) \leqslant x_2(t,.)$ in $L^2(Z)$ for all $t \in T$; i.e. $R(\cdot)$ is nondecreasing in K as claimed.

Claim 3. $\overline{R(K)}^{C(T,L^2(Z))}$ is compact and $\overline{R(K)}^{C(T,L^2(Z))} \subseteq K$.

Let $\{x_n\}_{n\geqslant 1} \subseteq R(K)$. Then $x_n = R(y_n)$, $y_n \in K$, $n \geqslant 1$. Because of hypothesis $H(f)$, by passing to a subsequence if necessary, we may assume that $\hat{f}(y_n) + My_n = u_n \xrightarrow{w} u$ in $L^q(T\times Z)$ as $n \longrightarrow \infty$ (as before $\hat{f}(y_n)(t,z) = f(t,z,y_n(t,z))$). We have

$$\left(\left(\frac{\partial x_n}{\partial t}, x_n\right)\right) + a(x_n, x_n) + M((x_n, x_n)) = ((u_n, x_n)), \quad n \geqslant 1.$$

Integrating by parts in $W_{pq}(T)$ and using hypothesis $H(a)$ (iv) we obtain

$$\hat{c}_2 \| Dx_n \|^p_{L^p(T\times Z)} \leqslant \| u_n \|_{L^q(T\times Z)} \| x_n \|_{L^p(T\times Z)} + \hat{\beta}_2 + \frac{1}{2} \| x_0 \|^2_{L^2(Z)}. \tag{15}$$

Recalling that $\| Dx_n \|_{L^p(T\times Z)}$ is equivalent to the norm $\| x_n \|_{L^p(T,W_0^{1,p}(Z))}$ we infer from (15) that $\{x_n\}_{n\geqslant 1}$ is bounded in $L^p(T, W_0^{1,p}(Z))$. Then directly from the equation we obtain that $\{\partial x_n/\partial t\}_{n\geqslant 1}$ is bounded in $L^q(T, W^{-1,q}(Z))$. Therefore $\{x_n\}_{n\geqslant 1}$ is bounded in $W_{pq}(T)$. Thus by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ as $n \to \infty$. Recalling from § 2 that $W_{pq}(T)$ is embedded compactly in $L^p(T\times Z)$, by passing to a further subsequence if necessary, we may assume that $x_n \to x$ in $L^p(T\times Z)$ and $x_n(t,.) \to x(t,.)$ in $L^2(Z)$ as $n \to \infty$ for all $t \in T\backslash N$ with $\lambda(N) = 0$ (here $\lambda(\cdot)$ is the Lebesgue measure on T). Let $\langle .,. \rangle$ denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$ and observe that the sequence $\{\langle \partial x_n/\partial t, x_n - x\rangle\}_{n\geqslant 1}$ is uniformly integrable. So given $\varepsilon > 0$, we can find $s \in T\backslash N$ such that

$$\int_s^b \left| \left\langle \frac{\partial x_n}{\partial t}, x_n(t) - x(t) \right\rangle \right| \mathrm{d}t \leqslant \varepsilon. \tag{16}$$

In what follows by $((.,.))_s$, $s \in (0,b]$, we will denote the duality brackets for the pair $(L^q([0,s], W^{-1,q}(Z)), L^p([0,s], W_0^{1,p}(Z)))$. Using the integration by parts formula for functions in $W_{pq}(T)$, we have

$$\left(\left(\frac{\partial x_n}{\partial t}, x_n - x\right)\right)_s = \frac{1}{2} \| x_n(s,.) - x(s,.) \|^2_{L^2(Z)} + \left(\left(\frac{\partial x}{\partial t}, x_n - x\right)\right)_s \to 0$$

$$\text{as } n \to \infty$$

(recall that $x_n(0,.) = x(0,.) = x_0(0)$). Then

$$\begin{aligned}\left(\left(\frac{\partial x_n}{\partial t}, x_n - x\right)\right) &= \int_0^b \left\langle \frac{\partial x_n}{\partial t}, x_n(t) - x(t) \right\rangle \mathrm{d}t \\ &= \left(\left(\frac{\partial x_n}{\partial t}, x_n - x\right)\right)_s + \int_s^b \left\langle \frac{\partial x_n}{\partial t}, x_n(t) - x(t) \right\rangle \mathrm{d}t\end{aligned}$$

and so using (16) above, we obtain

$$\underline{\lim} \left(\left(\frac{\partial x_n}{\partial t}, x_n - x\right)\right) \geqslant -\varepsilon. \tag{17}$$

Similarly we can have

$$\overline{\lim} \left(\left(\frac{\partial x_n}{\partial t}, x_n - x\right)\right) \leqslant \varepsilon. \tag{18}$$

Since $\varepsilon > 0$ was arbitrary, from (17) and (18) above we conclude that

$$\left(\left(\frac{\partial x_n}{\partial t}, x_n - x\right)\right) \to 0 \text{ as } n \to \infty.$$

But note that $((A(x_n), x_n - x)) = ((u_n, x_n - x)) - ((\partial x_n/\partial t, x_n - x)) \to 0$ as $n \to \infty$. Recall that the operator $A: L^p(T, W_0^{1,p}(Z)) \to L^q(T, W^{-1,q}(Z))$ is maximal monotone, so by proposition 1 of [3], it is also generalized pseudomonotone. Hence $A(x_n) \xrightarrow{w} A(x)$ in $L^q(T, W^{-1,q}(Z))$ and $((A(x_n), x_n)) \to ((A(x), x))$ as $n \to \infty$. So in the limit as $n \to \infty$, we have

$$\left\{\begin{array}{l} \dfrac{\partial x}{\partial t} - \displaystyle\sum_{k=1}^{N} D_k a_k(t, z, Dx(t, z)) + Mx = u(t, z)) \text{ on } T \times Z \\ x(0, z) = x_0(z) \text{ on } Z, x|_{T \times \Gamma} = 0 \end{array}\right\}$$

Therefore for all $t \in T$ we have

$$\frac{1}{2}\|x_n(t,.) - x(t,.)\|_{L^2(Z)}^2 + ((A(x_n) - A(x), x_n - x))_t = ((u_n - u, x_n - x))_t$$

so

$$\frac{1}{2}\|x_n(t,.) - x(t,.)\|_{L^2(Z)}^2 \leqslant \|u_n - u\|_{L^q(T \times Z)} \|x_n - x\|_{L^p(T \times Z)}$$

$$\leqslant M_1 \|x_n - x\|_{L^p(T \times Z)} \text{ for some } M_1 > 0.$$

Thus we conclude that

$$\sup_{t \in T} \|x_n(t,.) - x(t,.)\|_{L^2(Z)} \to 0 \text{ as } n \to \infty$$

and so $x_n \to x$ in $C(T, L^2(Z))$ as $n \to \infty$. This proves that $\overline{R(K)}^{C(T, L^2(Z))}$ is compact and $\overline{R(K)}^{C(T, L^2(Z))} \subseteq K$.

Now let $\{y_n\}_{n \geqslant 1} \subseteq K$ be a monotone sequence. Since $R(K) \subseteq K$ by claim 1 and $R(\cdot)$ is nondecreasing on K by claim 2, recalling that the positive cone in $L^2(Z)$ is regular (see [8], pp. 36 and 41), we have that $R(y_n)(t,.) \to x(t,.)$ in $L^2(Z)$ as $n \to \infty$ for all $t \in T$. Then by virtue of claim 3 we have that $R(y_n) = x_n \to x$ in $C(T, L^2(Z))$ as $n \to \infty$ and $x \in K$. Therefore we can apply theorem 2.1 with $V = C(T, L^2(Z))$, $\hat{K} = W_{pq}(T)$, $K = [a, b] = [\psi, \varphi]$ and $G = R$, to conclude that $R(\cdot)$ has a greatest (maximum) and a least (minimum) fixed point in K. Evidently these are the maximum and minimum solutions of (1) in $[\psi, \varphi]$. ∎

An interesting biproduct of the above proof is the compactness in $C(T, L^2(Z))$ of the solution set of (1) when the right hand side $f(t, z, x)$ is continuous in x. More specifically we assume the following:

$\mathbf{H(f)}_1$: $f: T \times Z \times R \to R$ is a function such that

(i) $(t, z) \to f(t, z, x)$ is measurable;
(ii) $x \to f(t, z, x)$ is continuous on $[\psi(t, z), \varphi(t, z)]$;
(iii) $|f(t, z, x)| \leqslant \vartheta(t, z)$ for almost all $(t, z) \in T \times Z$ and all $x \in [\psi(t, z), \varphi(t, z)]$, with $\vartheta \in L^q(T \times Z)$; and
(iv) there exists $M > 0$ such that for almost all $(t, z) \in T \times Z$, the function $x \to f(t, z, x) + Mx$ is nondecreasing.

COROLLARY 3.2 *If hypotheses $H(a)$ and $H(f)_1$ hold and $x_0 \in L^2(Z)$, then the solution set of* (1) *is nonempty, compact in $C(T, L^2(Z))$ and has a maximum and a minimum element.*

Remark 3.1. If the problem is semilinear (i.e. the partial differential operator is linear) and $f(t,z,.)$ is right continuous (resp. left continuous) on $[\psi(t,z), \varphi(t,z)]$, then the maximum (resp. the minimum) solution of the problem in $[\psi, \varphi]$ can be obtained in a constructive way via a monotone iterative process.

References

[1] Appell J and Zabrejko P, *Nonlinear superposition operators* (Cambridge: Cambridge Univ. Press) (1990)

[2] Barbu V, *Nonlinear semigroups and differential equations in Banach spaces* (The Netherlands: Leyden, Noordhoff Int. Publishing) (1976)

[3] Browder F and Hess P, Nonlinear mappings of monotone type in Banach spaces, *J. Funct. Anal.* **11** (1972) 251–294

[4] Cardinali T, Fiacca A and Papageorgiou N S, Extremal solutions for nonlinear parabolic problems with discontinuities, *Monatshefte für Mathematik* **124** (1997), 119–131

[5] Carl S and Heikkilä S, On a parabolic boundary value problem with discontinuous nonlinearity, *Nonlinear Anal.—TMA* **15** (1990) 1091–1095

[6] Gilbarg D and Trudinger N, *Elliptic partial differential equations of second order* (NY: Springer-Verlag) (1977)

[7] Heikkilä S and Laksmikantham V, *Monotne iterative techniques for discontinuous nonlinear differential operators* (NY: Marcel Dekker Inc.) (1994)

[8] Krasnoselskii M A, *Positive solutions of operator equations* (The Netherlands: Groningen, Noordhoff Ltd.) (1964)

[9] Zeidler E, *Nonlinear functional analysis and its applications II* (NY: Springer-Verlag) (1990)

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 189–207.

Homogenization of periodic optimal control problems via multi-scale convergence

S KESAVAN and M RAJESH

The Institute of Mathematical Sciences, C.I.T. Campus, Taramani, Chennai 600 113, India
E-mail: kesh@imsc.ernet.in, rajesh@imsc.ernet.in

MS received 23 February 1998

Abstract. The aim of this paper is to provide an alternate treatment of the homogenization of an optimal control problem in the framework of two-scale (multi-scale) convergence in the periodic case. The main advantage of this method is that we are able to show the convergence of cost functionals directly without going through the adjoint equation. We use a corrector result for the solution of the state equation to achieve this.

Keywords. Homogenization; optimal control; multi-scale convergence.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. We denote by $M(\alpha, \beta, \Omega)$ the set of all $N \times N$ matrices $A = ((a_{ij}))$ such that $a_{ij} \in L^\infty(\Omega)$ for $1 \leqslant i,j \leqslant N$ and

$$\alpha|\xi|^2 \leqslant a_{ij}\xi_i\xi_j \leqslant \beta|\xi|^2 a{\cdot}e(x), \quad 0 < \alpha < \beta \tag{1.1}$$

for all $\xi \in \mathbb{R}^N$. (Here and in the sequel we adopt the convention of summation over repeated indices). We now describe the optimal control problem.

Let $U_{ad} \subseteq L^2(\Omega)$ be a closed convex set. Let $f \in L^2(\Omega)$ be a fixed function. For $\theta \in U_{ad}$, we define the state variable $u = u(\theta) \in H_0^1(\Omega)$ as the (weak) solution of the following second order elliptic boundary value problem:

$$\begin{cases} -\operatorname{div}(A\nabla u) = f + \theta & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where $A \in M(\alpha, \beta, \Omega)$. We then consider the cost (or objective) functional defined by

$$J(\theta) = \frac{1}{2}\int_\Omega B\nabla u \nabla u \mathrm{d}x + \frac{N}{2}\int_\Omega \theta^2 \mathrm{d}x, \tag{1.3}$$

where, for $\theta \in U_{ad}$, u is the associated state variable (solution of (1.2)) and $B \in M(\alpha, \beta, \Omega)$ is a symmetric matrix. The problem is then to find $\theta^* \in U_{ad}$ (called the optimal control) such that

$$J(\theta^*) = \min_{\theta \in U_{ad}} J(\theta),$$

It is a standard result (cf. [7]) that there exists a unique optimal control θ^*.

We are now interested in the situation where we have a family of optimal control problems of the kind described above. More precisely, let $A_\varepsilon \in M(\alpha, \beta, \Omega)$ and $B_\varepsilon \in M(\alpha', \beta', \Omega)$ (with B_ε symmetric) where $\varepsilon > 0$ is a parameter which eventually tends to zero. Then we consider the problem: find $\theta_\varepsilon^* \in U_{ad}$ such that

$$J_\varepsilon(\theta_\varepsilon^*) = \min_{\theta \in U_{ad}} J_\varepsilon(\theta). \tag{1.4}$$

where

$$J_\varepsilon(\theta)=\frac{1}{2}\int_\Omega B_\varepsilon\nabla u_\varepsilon\nabla u_\varepsilon\mathrm{d}x+\frac{N}{2}\int_\Omega\theta^2\mathrm{d}x, \tag{1.5}$$

and $u_\varepsilon\in H_0^1(\Omega)$ is the state variable corresponding to $\theta\in U_{ad}$ and is the unique solution of the problem:

$$\begin{cases}-\operatorname{div}(A_\varepsilon\nabla u_\varepsilon)=f+\theta & \text{in } \Omega\\ \qquad u_\varepsilon=0 & \text{on } \partial\Omega.\end{cases} \tag{1.6}$$

It can be shown that $\{\theta_\varepsilon^*\}$, where θ_ε^* is the unique optimal control of the problem (1·4)–(1·6), is uniformly bounded in $L^2(\Omega)$ (with respect to ε) and so for a subsequence, $\theta_\varepsilon^*\rightharpoonup\theta_0^*$ weakly in $L^2(\Omega)$. The problem is to characterize θ_0^*. In particular, we wish to find the matrices A_0 and $B_\#$ with properties similar to those above so that θ_0^* is the optimal control of the corresponding problem.

This problem was first studied by Kesavan and Vanninathan [6] in the case when the coefficients A_ε and B_ε are periodic (see § 3 below for a precise description of this case). Kesavan and Saint Jean Paulin [4] solved it in the case of general coefficients and in a later paper (cf. [5]) extended it to the case when the domain Ω is replaced by a 'perforated domain Ω_ε'.

In all the papers cited above, the energy method in homogenization theory was used. Further the adjoint state variable $p_\varepsilon\in H_0^1(\Omega)$ was introduced via the equation

$$\begin{cases}\operatorname{div}({}^tA_\varepsilon\nabla p_\varepsilon-B_\varepsilon\nabla u_\varepsilon)=0 & \text{in } \Omega\\ \qquad p_\varepsilon=0 & \text{on } \partial\Omega\end{cases} \tag{1.7}$$

(with additional Neumann condition on the holes in case of perforated domains). The system (1.6)–(1.7) was first homogenized and from this the limit matrices A_0 and $B_\#$ were identified.

In this paper we restrict our attention to the periodic case. As technical device we use the notion of 2-scale convergence developed by Nguetseng [8] and Allaire [1]. We are then able to directly obtain the matrix $B_\#$ without the necessity of introducing the adjoint problem.

Given the problem

$$\begin{cases}-\operatorname{div}(A_\varepsilon\nabla w_\varepsilon)=f & \text{in } \Omega\\ \qquad w_\varepsilon=0 & \text{on } \partial\Omega\end{cases} \tag{1.8}$$

where the A_ε are periodic, we are directly able to calculate the limit of the integral

$$\int_\Omega B_\varepsilon\nabla w_\varepsilon\nabla w_\varepsilon\mathrm{d}x.$$

Once this is done, the procedure outlined by Kesavan and Saint Jean Paulin [5] establishes the convergence of the optimal control.

Of course, the method of 2-scale convergence necessitates the assumption of some regularity of the coefficients A_ε.

We also prove, along the way, some slightly improved versions of results on 2-scale convergence compared with those of Allaire [1]. We are able to directly deal with the periodically perforated domain and we prove some corrector results for the solution of the analogue of (1.8) in that case.

Finally, we are able to easily extend our results to the multi-scale case, i.e. where there are several (well separated) scales of periodicity in the coefficients using the results of multi-scale convergence of Allaire and Briane [2].

2. 2-Scale convergence

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set. Let Y denote the unit cell $[0,1]^N$ in $\mathbb{R}^N$. In this paper we use the symbol $\|\cdot\|_{p,\Omega}$ to denote the L^p norm of a function defined on Ω.

For $\varepsilon > 0$, and g a function defined on $\Omega \times Y$, we define an oscillating function $g(x,(x/\varepsilon))$ as follows. Cover $\mathbb{R}^N$ with translates of the ε-cell, εY. Then any $x \in \Omega$ falls in a translate of the cell εY and hence corresponds to a unique y in Y. Define $g(x,(x/\varepsilon))$ to be the value of g at (x,y). Here and in the sequel, we denote a function that is Y-periodic by the subscript #.

DEFINITION 2.1

A sequence $\{u_\varepsilon\}$ of functions in $L^2(\Omega)$, where ε is a parameter which tends to zero, is said to 2-scale converge to a function $u_0 \in L^2(\Omega \times Y)$ if

$$\int_\Omega u_\varepsilon \phi\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x \to \int_\Omega\int_Y u_0(x,y)\phi(x,y)\mathrm{d}y\mathrm{d}x \quad \text{for all} \quad \phi \in D(\Omega, C_{\#}^{\infty}(Y)). \tag{2.1}$$

We write $u_\varepsilon \xrightarrow{2-s} u_0(x,y)$. □

The relevance of this definition stems from the following result.

Theorem 2.1. [1] *Every bounded sequence in $L^2(\Omega)$ has a 2-scale convergent subsequence.* □

Remark 2.1.

(1) For any 2-scale convergent sequence its 2-scale limit is unique.
(2) If $\phi(x,y)$ is 'smooth' (for instance if it belongs to one of the spaces listed below in Remark 2.2), then $\phi(x,(x/\varepsilon)) \xrightarrow{2-s} \phi(x,y)$.
(3) If $u_\varepsilon \to u$ strongly in $L^2(\Omega)$, then $u_\varepsilon \xrightarrow{2-s} u(x)$.
(4) If $u_\varepsilon \xrightarrow{2-s} u_0(x,y)$, then $u_\varepsilon \to \int_Y u_0(x,y)$ weakly in $L^2(\Omega)$ (take test functions depending on x alone).
(5) As a result of the previous remark and the uniform boundedness principle, any 2-scale convergent sequence is bounded.
(6) Supposing that u_ε admits an asymptotic expansion,

$$u_\varepsilon = u_0\left(x,\frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x,\frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x,\frac{x}{\varepsilon}\right) + \cdots, \tag{2.2}$$

where $u_i(x,y)$ are assumed to be 'smooth', then $u_\varepsilon \xrightarrow{2-s} u_0(x,y)$. So the 2-scale limit gives the first term in the asymptotic expansion of u_ε, when the expansion is valid. □

DEFINITION 2.2

A measurable function $\psi:\Omega \times Y \to \mathbb{R}$ *which is periodic in the variable y is said to be admissible if*

$$\int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right)^2 \mathrm{d}x \to \int_\Omega \int_Y \psi(x,y)^2 \mathrm{d}y\mathrm{d}x. \tag{2.3}$$

More generally, let $\{u_\varepsilon\}$ *be a sequence which 2-scale converges to* $u_0(x,y)$. *It is said to be admissible if*

$$\int_\Omega (u_\varepsilon)^2 \mathrm{d}x \to \int_\Omega \int_Y u_0(x,y)^2 \mathrm{d}y\mathrm{d}x. \tag{2.4}$$

□

Remark 2.2 Though the most general condition under which $\psi(x,y)$ is admissible is not known, it is known that if ψ belongs to one of the space $L^2(\Omega, C_\#(Y))$, $C_c(\Omega, L^\infty_\#(Y))$ or $C(\bar{\Omega}, L^\infty_\#(Y))$ then it is admissible (cf. [1]). □

Theorem 2.2. *Let* $u_\varepsilon \xrightarrow{2-s} u_0(x,y)$ *and assume that* $\{u_\varepsilon\}$ *is an admissible sequence. If* v_ε *is any sequence such that* $v_\varepsilon \xrightarrow{2-s} v_0(x,y)$ *then,*

$$u_\varepsilon v_\varepsilon \to \int_Y u_0(x,y)v_0(x,y)\mathrm{d}y \text{ in } D'(\Omega) \tag{2.5}$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon v_\varepsilon \mathrm{d}x = \int_\Omega\int_Y u_0(x,y)v_0(x,y)\mathrm{d}y\mathrm{d}x. \tag{2.6}$$

Further, if $u_0(x,(x/\varepsilon)) \xrightarrow{2-s} u_0(x,y)$ *and* u_0 *is an admissible function then,*

$$\lim_{\varepsilon\to 0}\left\| u_\varepsilon - u_0\left(x,\frac{x}{\varepsilon}\right)\right\|_{2,\Omega} = 0. \tag{2.7}$$

□

Remark 2.3. (1) In fact, except for (2.6), this result is proved in [1]. The same proof can be easily adapted to give (2.6) as well. (2) The hypothesis for (2.7) is slightly more general than saying (cf. [1]) $\psi \in L^2(\Omega, C_\#(Y))$. □

We now prove a result which will be used repeatedly in the sequel.

Theorem 2.3. *Suppose that* $u_\varepsilon \xrightarrow{2-s} u_0(x,y)$ *and* $\phi \in C(\bar{\Omega}, L^\infty_\#(Y))$ *then,*

$$u_\varepsilon \phi\left(x,\frac{x}{\varepsilon}\right) \xrightarrow{2-s} u_0(x,y)\phi_0(x,y). \tag{2.8}$$

Proof. Since $\|\phi(x,(x/\varepsilon))\|_{\infty,\Omega} \leqslant \|\phi(x,y)\|_{\infty,\Omega\times Y}$ and $\{u_\varepsilon\}$ is bounded in $L^2(\Omega)$ we have,

$$\left\|u_\varepsilon\phi\left(x,\frac{x}{\varepsilon}\right)\right\|_{2,\Omega} \leqslant c\|u_\varepsilon\|_{2,\Omega} \leqslant c \text{ for all } \varepsilon \text{ (where } c \text{ is a generic constant).}$$

By Theorem 2.1, for every subsequence of $\{u_\varepsilon\phi(x,(x/\varepsilon))\}$ there is a further subsequence (which we continue to index by ε for simplicity) and a function $u(x,y)$ such that

$u_\varepsilon \phi(x,(x/\varepsilon))$ 2-scale converges to $u(x,y)$. We will show that $u(x,y) = u_0(x,y)\phi(x,y)$. Since this limit is independent of the subsequence chosen, it shows that the entire sequence 2-scale converges to this limit.

To prove the claim made above, let $\psi \in D(\Omega, C^\infty_\#(Y))$. Then we have by definition,

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon \phi\left(x,\frac{x}{\varepsilon}\right)\psi\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x = \int_\Omega\int_Y u\psi\,\mathrm{d}y\mathrm{d}x.$$

On the other hand, $\phi\psi \in C(\bar{\Omega}, L^\infty_\#(Y))$ and so $\phi(x,(x/\varepsilon))\psi(x,(x/\varepsilon)) \xrightarrow{2-s} \phi(x,y)\psi(x,y)$ and $\phi\psi$ is an admissible function (cf. Remark 2.2). Therefore by Theorem 2.2 we get,

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon \phi\left(x,\frac{x}{\varepsilon}\right)\psi\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x = \int_\Omega\int_Y u_0(x,y)\phi(x,y)\psi(x,y)\mathrm{d}x\mathrm{d}y$$

as u_0 is the 2-scale limit of u_ε. From the above and the density of $D(\Omega, C^\infty_\#(Y))$ functions in $L^2(\Omega \times Y)$ we conclude that $u = u_0\phi$. This completes the proof. □

COROLLARY 2.1

Let u_ε, u_0 and ϕ be as in Theorem 2.3. Let $\{v_\varepsilon\}$ be an admissible sequence which 2-scale converges to $v_0(x,y)$. Then,

$$\int_\Omega u_\varepsilon \phi\left(x,\frac{x}{\varepsilon}\right)v_\varepsilon \mathrm{d}x \to \int_\Omega\int_Y u_0\phi v_0 \mathrm{d}x\mathrm{d}y. \tag{2.9}$$

Proof. By Theorem 2.3, $u_\varepsilon\phi(x,(x/\varepsilon)) \xrightarrow{2-s} u_0(x,y)\phi(x,y)$. Also, $v_\varepsilon \xrightarrow{2-s} v(x,y)$ and $\{v_\varepsilon\}$ is an admissible sequence. Therefore, (2.9) follows from Theorem 2.2. □

Theorem 2.4. [1] *Let $\{u_\varepsilon\}$ be a bounded sequence in $H^1_0(\Omega)$ that weakly converges to a function u in $H^1_0(\Omega)$. Then there exists a function $u_1 \in L^2(\Omega, H^1_\#(Y)/\mathbb{R})$ such that (for a subsequence)*

$$u_\varepsilon \xrightarrow{2-s} u(x) \quad \text{and,}$$
$$\nabla u_\varepsilon \xrightarrow{2-s} \nabla_x u + \nabla_y u_1(x,y).$$ □

Remark 2.4. (1) $H^1_\#(Y)$ is the space of functions in $H^1(Y)$ which have been extended by Y-periodicity to $\mathbb{R}^N$.(2) In case u_ε has an asympotic expansion then it would be of the form

$$u_\varepsilon(x) = u_0(x) + u_1\left(x,\frac{x}{\varepsilon}\right) + \cdots.$$ □

3. Convergence of cost functionals

Let Ω be a bounded open set in $\mathbb{R}^N$. We obtain a periodically perforated domain Ω_ε by removing from Ω a set of periodically distributed holes, T_ε. i.e. $\Omega_\varepsilon = \Omega\setminus\overline{T_\varepsilon}$, where T_ε is defined as follows.

Let T be an open subset of the unit cell $Y = [0,1]^N$ with Lipschitz boundary. Set,

$$T_\varepsilon = \bigcup_{k\in\mathbb{Z}^N} \varepsilon(k+T).$$

We denote the 'material part' of the unit cell by Y^*, i.e. $Y^* = Y\setminus\bar{T}$.

The boundary of Ω_ε has two parts-one comprises the union of boundaries of holes strictly contained in Ω, denoted by $\partial_{\text{int}}\Omega_\varepsilon$.

$$\partial_{\text{int}}\Omega_\varepsilon = \bigcup_{\mathbf{k}\in\mathbb{Z}^N} \{\partial_\varepsilon(\mathbf{k}+T): \varepsilon(\mathbf{k}+\bar{T}) \subset \Omega\}.$$

The second part is the exterior boundary,

$$\partial_{\text{ext}}\Omega_\varepsilon = \partial\Omega_\varepsilon \setminus \partial_{\text{int}}\Omega_\varepsilon.$$

We make the following assumptions:

$$\Omega_\varepsilon \text{ is a connected set.} \tag{3.1}$$

Let A be the $N \times N$ matrix $A = ((a_{ij}(x,y)))$ in $M(\alpha,\beta,\Omega \times Y)$ such that

$$a_{ij} \in C(\bar{\Omega}, L^\infty_\#(Y)). \tag{3.2}$$

Consider a sequence $\{f_\varepsilon\}$ *in* $L^2(\Omega_\varepsilon)$ and a function $f \in L^2(\Omega)$ such that,

$$\tilde{f}_\varepsilon \to \lambda f \text{ weakly in } L^2(\Omega), \tag{3.3}$$

where $\lambda = |Y^*|$, the volume of the material part of the unit cell and the $\sim$ denotes the extension by zero outside Ω_ε.

We consider the following problem posed in Ω_ε with a Neumann condition on interior holes:

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f_\varepsilon & \text{in } \Omega_\varepsilon \\ A_\varepsilon \nabla u_\varepsilon \cdot n_\varepsilon = 0 & \text{on } \partial_{\text{int}}\Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial_{\text{ext}}\Omega_\varepsilon. \end{cases} \tag{3.4}$$

Introduce the space,

$$V_\varepsilon = \{u \in H^1_0(\Omega_\varepsilon): u = 0 \text{ on } \partial_{\text{ext}}\Omega_\varepsilon\}$$

and the bilinear form, $a_\varepsilon: V_\varepsilon \times V_\varepsilon \to \mathbb{R}$

$$a_\varepsilon(u,v) = \int_\Omega A_\varepsilon \nabla u \nabla u \, dx,$$

where A_ε is the matrix $A(x,(x/\varepsilon))$.

Then (3.4) admits the weak formulation,

$$\begin{cases} \text{Find } u_\varepsilon \in V_\varepsilon \text{ such that} \\ a_\varepsilon(u_\varepsilon, v) = (f_\varepsilon, v) \text{ for all } v \in V_\varepsilon. \end{cases} \tag{3.5}$$

It is known that a_ε is coercive and hence (3.5) admits a unique solution u_ε in V_ε. It is also known that $\{\tilde{u}_\varepsilon\}$ is a bounded sequence in $L^2(\Omega)$(cf. [1]). Hence assume for a subsequence that

$$\tilde{u}_\varepsilon \to \lambda u \text{ weakly in } L^2(\Omega) \text{ for a function } u \in L^2(\Omega). \tag{3.6}$$

Theorem 3.1. *Let u_ε be the solution of* (3.5). *Then for a subsequence*

$$\begin{cases} \tilde{u}_\varepsilon \xrightarrow{2-s} \chi(y)u(x) \\ \widetilde{\nabla u_\varepsilon} \xrightarrow{2-s} \chi(y)(\nabla_x u(x) + \nabla_y u_1(x,y)) \end{cases} \tag{3.7}$$

where $u \in H^1_0(\Omega)$, $u_1 \in L^2(\Omega, C^1_\#(Y)/\mathbb{R})$ and χ is the characteristic function of the set Y^*.

In fact, if we choose u_1 such that $\int_{Y^*} u_1(x,y)\mathrm{d}y = 0$ then, (u, u_1) is the unique solution in $H_0^1(\Omega) \times L^2(\Omega, C_\#^1(Y)/\mathbb{R})$ of the 2-scale homogenized problem

$$\begin{cases} -\operatorname{div}_y(A(x,y)(\nabla_x u + \nabla_y u_1(x,y)) = 0 & \text{in } \Omega \times Y^* \\ A(x,y)(\nabla_x u + \nabla_y u_1(x,y))\cdot n_y = 0 & \text{on } \partial Y^* \setminus \partial Y \\ -\operatorname{div}_x(\int_Y \chi(y) A(x,y)(\nabla_x u + \nabla_y u_1(x,y))\mathrm{d}y) = \lambda f & \text{in } \Omega. \end{cases} \tag{3.8}$$

□

Remark 3.1. (1) This theorem is proved in [1] for $f_\varepsilon \equiv f$ in Ω_ε. The same proof can be adapted to prove Theorem 3.1 where the right hand side in equation (3.5) is a sequence $\{f_\varepsilon\}$ such that $\tilde{f}_\varepsilon \to \lambda f$. (2) The extra regularity of u_1 comes from the smoothness of the coefficients $a_{ij}(x,y)$. (3) The equations (3.8) may be decoupled by setting

$$u_1(x,y) = \frac{\partial u}{\partial x_i} X^i(x,y),$$

where, $X^i(x,y)$ is the solution of the problem:

$$\begin{cases} -\operatorname{div}_y(A(x,y)(e^i + \nabla X^i(x,y))) = 0 & \text{in } Y^* \text{ a.e. } x \\ \int_{Y^*} X^i(x,y)\mathrm{d}y = 0 & \text{a.e. } x \\ y \mapsto \;\; X^i(x,y) \text{ is } Y\text{-periodic} & \end{cases} \tag{3.9}$$

for $i = 1, 2, \ldots, N$ and u is the solution of

$$\begin{cases} -\operatorname{div}_x(A_0(x)\nabla u) = \lambda f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.10}$$

where the matrix A_0 given by,

$$(A_0(x))_{ij} = \int_Y \chi(y)\left(a_{ij}(x,y) + a_{ik}(x,y)\frac{\partial X^i}{\partial_{yk}}(x,y)\right)\mathrm{d}y,$$

is the H_0-limit of the sequence $\{A_\varepsilon\}$. □

We now prove a corrector result. First we need the following preliminary result.

Lemma 3.1. *Let u_ε be the solution of* (3.5) *and* (u, u_1) *be as in Theorem* 3.1. *Then* $\{\chi_\varepsilon(\partial u/\partial x_i + \partial u_1/\partial y_i(x,(x/\varepsilon)))\}$ *is an admissible sequence for* $i = 1, 2, \ldots N$, *where* χ_ε *is the characteristic function of the set* Ω_ε.

Proof. We have, by Theorem 2.3,

$$\chi_\varepsilon\left(\frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}\left(x, \frac{x}{\varepsilon}\right)\right) \xrightarrow{2-s} \left(\frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}(x,y)\right)\chi(y).$$

Since $\partial u_1/\partial y_i(x,(x/\varepsilon))$ vanishes on the holes, we can write

$$\begin{aligned} \int_\Omega \chi_\varepsilon\left(\frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\right)^2 \mathrm{d}x &= \int_\Omega \left(\chi_\varepsilon\frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\right)^2 \mathrm{d}x \\ &= \int_\Omega \chi_\varepsilon\left(\frac{\partial u}{\partial x_i}\right)^2 \mathrm{d}x + 2\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x \\ &\quad + \int_\Omega \left(\frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\right)^2 \mathrm{d}x. \end{aligned}$$

Now,

$$\int_\Omega \chi_\varepsilon\left(\frac{\partial u}{\partial x_i}\right)^2 \mathrm{d}x \to \int_\Omega\int_Y \chi(y)\left(\frac{\partial u}{\partial x_i}\right)^2 \mathrm{d}y\mathrm{d}x$$

since $\chi_\varepsilon \rightharpoonup S_Y\chi(y)\mathrm{d}y$ in $L^\infty(\Omega)$ weak * and $(\partial u/\partial x_i)^2 \in L^1(Y)$. Next,

$$\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x \longrightarrow \int_\Omega\left(\int_Y \frac{\partial u_1}{\partial y_i}(x,y)\mathrm{d}y\right)\frac{\partial u}{\partial x_i}\mathrm{d}x$$

$$= \int_\Omega\int_Y \chi(y)\frac{\partial u}{\partial x_i}\frac{\partial u_1}{\partial y_i}(x,y)\mathrm{d}y\,\mathrm{d}x$$

since $\partial u_1/\partial y_i(x,(x/\varepsilon)) \to \int_Y \partial u_1/\partial y_i(x,y)\mathrm{d}y$ weakly in $L^2(\Omega)$ and $\partial u_1/\partial y_i(x,y) = \chi(y)$ $\partial u_1/\partial y_i(x,y)$. Finally,

$$\int_\Omega \frac{\partial u_1}{\partial y_i}^2\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x \longrightarrow \int_\Omega\int_Y \left(\frac{\partial u_1}{\partial y_i}(x,y)\right)^2 \mathrm{d}y\mathrm{d}x \text{ (since } u_1 \text{ is smooth.)}$$

$$= \int_\Omega\int_Y \chi(y)\left(\frac{\partial u_1}{\partial y_i}(x,y)\right)^2 \mathrm{d}y\mathrm{d}x.$$

Thus

$$\lim_{\varepsilon\to 0}\int_\Omega \chi_\varepsilon\left(\frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\right)^2 \mathrm{d}x = \int_\Omega\int_Y \left\{\chi(y)\left(\frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}(x,y)\right)\right\}^2 \mathrm{d}y\mathrm{d}x$$

which proves the lemma. □

We are now in a position to prove the following corrector result.

Theorem 3.2. *Let u_ε be the solution of* (3.5). *Then*

$$\lim_{\varepsilon\to 0}\left\|\widetilde{\frac{\partial u_\varepsilon}{\partial x_i}} - \frac{\partial u}{\partial x_i}\chi_\varepsilon - \frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\chi_\varepsilon\right\|_{2,\Omega} = 0 \text{ for } i = 1,2,\ldots N \tag{3.11}$$

where (u, u_1) are as in Theorem 3.1.

Proof. Let

$$r_\varepsilon^i = \widetilde{\frac{\partial u_\varepsilon}{\partial x_i}} - \frac{\partial u}{\partial x_i}\chi_\varepsilon - \frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\chi_\varepsilon.$$

Then,

$$\alpha\sum_{i=1}^N \left\|r_\varepsilon^i\right\|_{2,\Omega}^2 \leqslant$$

$$\int_{\Omega_\varepsilon} a_{ij}\left(x,\frac{x}{\varepsilon}\right)\left(\frac{\partial u_\varepsilon}{\partial x_j} - \frac{\partial u}{\partial x_j} - \frac{\partial u_1}{\partial y_j}\left(x,\frac{x}{\varepsilon}\right)\right)\left(\frac{\partial u_\varepsilon}{\partial x_i} - \frac{\partial u}{\partial x_i} - \frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\right)\mathrm{d}x.$$

Therefore, using equation (3.4), this can be written as

$$\alpha\sum_{i=1}^N \|r_\varepsilon^i\|_{2,\Omega}^2 \leqslant \int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon \mathrm{d}x - \int_\Omega a_{ij}\left(x,\frac{x}{\varepsilon}\right)\widetilde{\frac{\partial u_\varepsilon}{\partial x_j}}\left(\frac{\partial u}{\partial x_i}\chi_\varepsilon + \frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\chi_\varepsilon\right)\mathrm{d}x$$

$$-\int_\Omega a_{ij}\left(x,\frac{x}{\varepsilon}\right)\left(\frac{\partial u}{\partial x_j}\chi_\varepsilon+\frac{\partial u_1}{\partial y_j}\left(x,\frac{x}{\varepsilon}\right)\chi_\varepsilon\right)\widetilde{\frac{\partial u_\varepsilon}{\partial x_i}}\mathrm{d}x+\int_\Omega a_{ij}\left(x,\frac{x}{\varepsilon}\right)$$
$$\times\left(\frac{\partial u}{\partial x_j}\chi_\varepsilon+\frac{\partial u_1}{\partial y_j}\left(x,\frac{x}{\varepsilon}\right)\chi_\varepsilon\right)\left(\frac{\partial u}{\partial x_i}\chi_\varepsilon+\frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\chi_\varepsilon\right)\mathrm{d}x. \tag{3.12}$$

We make the following observations,

$$\widetilde{\frac{\partial u_\varepsilon}{\partial x_i}}\xrightarrow{2-s}\chi(y)\left(\frac{\partial u}{\partial x_i}+\frac{\partial u_1}{\partial y_i}(x,y)\right)$$

$$a_{ij}(x,y)\in C(\bar{\Omega},L^\infty_\#(Y)),$$

$$\left(\frac{\partial u}{\partial x_i}\chi_\varepsilon+\frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\chi_\varepsilon\right)\xrightarrow{2-s}\chi(y)\left(\frac{\partial u}{\partial x_i}+\frac{\partial u_1}{\partial y_i}(x,y)\right).$$

Further by lemma 3.1, $\{(\partial u/\partial x_i)\chi_\varepsilon+(\partial u_1/\partial y_i)(x,(x/\varepsilon)\chi_\varepsilon)\}$ is an admissible sequence.

So by Corollary 2.1 and from the observations made above it follows that

$$\int_\Omega a_{ij}\left(x,\frac{x}{\varepsilon}\right)\widetilde{\frac{\partial u_\varepsilon}{\partial x_j}}\left(\frac{\partial u}{\partial x_i}\chi_\varepsilon+\frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\chi_\varepsilon\right)\mathrm{d}x\to$$
$$\int_\Omega\int_Y\chi(y)A(x,y)(\nabla_x u+\nabla_y u_1(x,y))(\nabla_x u+\nabla_y u_1(x,y))\mathrm{d}y\mathrm{d}x,$$

$$\int_\Omega a_{ij}\left(x,\frac{x}{\varepsilon}\right)\left(\frac{\partial u}{\partial x_j}\chi_\varepsilon+\frac{\partial u_1}{\partial y_j}\left(x,\frac{x}{\varepsilon}\right)\chi_\varepsilon\right)\widetilde{\frac{\partial u_\varepsilon}{\partial x_i}}\mathrm{d}x\to$$
$$\int_\Omega\int_Y\chi(y)A(x,y)(\nabla_x u+\nabla_y u_1(x,y))(\nabla_x u+\nabla_y u_1(x,y))\mathrm{d}y\mathrm{d}x$$

and

$$\int_\Omega a_{ij}\left(x,\frac{x}{\varepsilon}\right)\left(\frac{\partial u}{\partial x_j}\chi_\varepsilon+\frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\chi_\varepsilon\right)\left(\frac{\partial u}{\partial x_i}\chi_\varepsilon+\frac{\partial u_1}{\partial y_i}\left(x,\frac{x}{\varepsilon}\right)\chi_\varepsilon\right)\mathrm{d}x\to$$
$$\int_\Omega\int_Y\chi(y)A(x,y)(\nabla_x u+\nabla_y u_1(x,y))(\nabla_x u+\nabla_y u_1(x,y))\mathrm{d}y\mathrm{d}x.$$

Summing over the limits of the second, third and fourth terms on right hand side of equation (3.12) we get

$$-\int_\Omega\int_Y\chi(y)A(x,y)(\nabla_x u+\nabla_y u_1(x,y))(\nabla_x u+\nabla_y u_1(x,y))\mathrm{d}y\mathrm{d}x$$

which, by (3.8), is $\int_\Omega\lambda fu\mathrm{d}x$. Now notice that

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}\leqslant c \text{ (independent of } \varepsilon)$$
$$u_\varepsilon=0 \text{ on } \partial_{\mathrm{ext}}\Omega_\varepsilon$$
$$\tilde{u}_\varepsilon\rightharpoonup\lambda u \text{ and,}$$
$$\tilde{f}_\varepsilon\rightharpoonup\lambda f.$$

So by a strong compactness result of Allaire and Nandakumar [3],

$$\int_{\Omega_\varepsilon}f_\varepsilon u_\varepsilon\mathrm{d}x\to\int_\Omega\lambda fu\mathrm{d}x.$$

Therefore, it follows from (3.12) that $\|r_\varepsilon^i\|_{2,\Omega} \to 0$ as $\varepsilon \to 0$, which completes the proof. □

Remark 3.4. If f_ε is the restriction of a given function f in $L^2(\Omega)$ to Ω_ε for all ε, then

$$\int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon \mathrm{d}x = \int_\Omega f \tilde{u}_\varepsilon \mathrm{d}x \to \int_\Omega f \lambda u \mathrm{d}x,$$

since $\tilde{u}_\varepsilon \rightharpoonup \lambda u$ weakly in $L^2(\Omega)$. In this case we do not use the compactness result of Allaire and Nandakumar. □

COROLLARY 3.1

$\widetilde{\partial u_\varepsilon / \partial x_i}$ *is an admissible sequence for all* $i = 1, 2, \ldots, N$. *i.e.*,

$$\lim_{\varepsilon \to 0} \int_\Omega \left(\widetilde{\frac{\partial u_\varepsilon}{\partial x_i}}\right)^2 \mathrm{d}x = \int_\Omega \int_Y \chi(y) \left(\frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}\right)^2 \mathrm{d}y \mathrm{d}x.$$

Proof. We have

$$\int_\Omega \left(\widetilde{\frac{\partial u_\varepsilon}{\partial x_i}}\right)^2 \mathrm{d}x = \int_\Omega \left(r_\varepsilon^i + \frac{\partial u}{\partial x_i}\chi_\varepsilon + \frac{\partial u_1}{\partial y_i}\left(x, \frac{x}{\varepsilon}\right)\chi_\varepsilon\right)^2 \mathrm{d}x.$$

So,

$$\int_\Omega \left(\widetilde{\frac{\partial u_\varepsilon}{\partial x_i}}\right)^2 \mathrm{d}x = \int_\Omega (r_\varepsilon^i)^2 \mathrm{d}x + 2\int_\Omega r_\varepsilon^i \left(\frac{\partial u}{\partial x_i}\chi_\varepsilon + \frac{\partial u_1}{\partial y_i}\left(x, \frac{x}{\varepsilon}\right)\chi_\varepsilon\right) \mathrm{d}x$$
$$+ \int_\Omega \left(\frac{\partial u}{\partial x_i}\chi_\varepsilon + \frac{\partial u_1}{\partial y_i}\left(x, \frac{x}{\varepsilon}\right)\right)^2 \mathrm{d}x. \tag{3.13}$$

By Theorem 3.2, $r_\varepsilon^i \to 0$ in $L^2(\Omega)$ strongly. Also, $\{\chi_\varepsilon(\partial u/\partial x_i + \partial u_1/\partial y_i(x, (x/\varepsilon))\}$ is a bounded sequence in $L^2(\Omega)$. So the first two terms in the right hand side of (3.13) converge to zero as $\varepsilon \to 0$. Therefore,

$$\lim_{\varepsilon \to 0} \int_\Omega \left(\widetilde{\frac{\partial u_\varepsilon}{\partial x_i}}\right)^2 \mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega \left(\frac{\partial u}{\partial x_i}\chi_\varepsilon + \frac{\partial u_1}{\partial y_i}\left(x, \frac{x}{\varepsilon}\right)\right)^2 \mathrm{d}x$$
$$= \int_\Omega \int_Y \left(\frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}(x, y)\right)^2 \chi(y) \mathrm{d}y \mathrm{d}x \text{ (by lemma 3.1)},$$

i.e., $\{\widetilde{\partial u_\varepsilon / \partial x_i}\}$ is an admissible sequence. □

For u_ε as above i.e. solving (3.5) we have the following result on the convergence of cost functionals.

Let B be the $N \times N$ matrix $((b_{ij}(x, y)))$. Assume that $b_{ij} \in C(\bar{\Omega}, L_\#^\infty(Y))$ for all i, j. Denote the matrix $(b_{ij}(x, (x/\varepsilon)))$ by B_ε.

Theorem 3.3. *With the above assumptions on* u_ε *and* B *we have,*

$$\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \mathrm{d}x \to$$
$$\int_\Omega \int_Y \chi(y) B(x, y)(\nabla_x u + \nabla_y u_1(x, y))(\nabla_x u + \nabla_y u_1(x, y)) \mathrm{d}y \mathrm{d}x.$$

Proof. We may write

$$\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \mathrm{d}x = \int_\Omega b_{ij}\left(x, \frac{x}{\varepsilon}\right) \widetilde{\frac{\partial u_\varepsilon}{\partial x_j}} \widetilde{\frac{\partial u_\varepsilon}{\partial x_i}} \mathrm{d}x.$$

Note the following,

$$b_{ij} \in C(\bar{\Omega}, L^\infty_\#(Y))$$

$$\widetilde{\frac{\partial u_\varepsilon}{\partial x_i}} \xrightarrow{2-s} \chi(y)\left(\frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}(x,y)\right)$$

$$\left\{\widetilde{\frac{\partial u_\varepsilon}{\partial x_i}}\right\} \text{ is an admissible sequence.}$$

So, by Corollary 2.1,

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega b_{ij}\left(x, \frac{x}{\varepsilon}\right) \widetilde{\frac{\partial u_\varepsilon}{\partial x_j}} \widetilde{\frac{\partial u_\varepsilon}{\partial x_i}} \mathrm{d}x$$

$$= \int_\Omega \int_Y b_{ij}(x,y)\left(\frac{\partial u}{\partial x_j} + \frac{\partial u_1}{\partial y_j}(x,y)\right)\chi(y)\left(\frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}(x,y)\right)\chi(y)\mathrm{d}y\mathrm{d}x$$

$$= \int_\Omega \int_Y \chi(y) B(x,y)(\nabla_x u + \nabla_y u_1(x,y))(\nabla_x u + \nabla_y u_1(x,y))\mathrm{d}y\mathrm{d}x. \qquad \square$$

Remarks 3.5. (1) We can write, when B is symmetric,

$$\int_\Omega \int_Y \chi(y) B(x,y)(\nabla_x u + \nabla_y u_1(x,y))(\nabla_x u + \nabla_y u_1(x,y))\mathrm{d}y\mathrm{d}x$$

$$= \int_\Omega B_\# \nabla u \nabla u \mathrm{d}x$$

where,

$$(B_\#)_{ij} = (B_0)_{ij} + \int_Y \chi(y) B \nabla_y (X^i - Y^i) \nabla_y (X^j - Y^j) \mathrm{d}y.$$

Here, Y^i is the solution of the ith cell problem for B viz.,

$$\begin{aligned} -\operatorname{div}_y(B(x,y)(e^i + \nabla_y Y^i(x,y))) &= 0 \quad \text{in } Y^* \\ B(x,y)(e^i + \nabla_y Y^i(x,y))\cdot n &= 0 \quad \text{on } \partial Y^* \backslash \partial Y \\ \textstyle\int_{Y^*} Y^i(x,y)\mathrm{d}y &= 0 \\ y \mapsto Y^i(x,y) &\text{ is Y-periodic} \end{aligned}$$

and B_0 is the H_0 limit of the matrices B_ε and is given by the formula

$$(B_0)_{ij} = \int_Y \chi(y)\left(b_{ij} + b_{ik}\frac{\partial Y^j}{\partial y_k}\right)\mathrm{d}y.$$

(2) If there are no holes then $\chi(y) \equiv 1$. Hence we obtain the same formula for $B_\#$ as in [6, 4].

(3) In the above mentioned papers, the existence of $B_\#$ was proved in the context of an optimal control problem. The authors obtained the matrix $B_\#$ by introducing the adjoint state variable and analysing the corresponding adjoint equation. While their

analysis works for general coefficients, we have obtained the cost functional in the periodic case directly, without involving the adjoint problem.
(4) Once we have the convergence of cost functionals we can complete the study of the optimal control problem as in [5]. □

4. Reiterated homogenization

In physical problems involving more than one microscopic scale it is useful to have the notion of multi-scale convergence. The definition and main results of multi-scale convergence as introduced by Allaire and Briane [2] are recalled. We later apply this method to obtain the limit of quadratic functionals, thereby generalizing the results obtained in § 3 to a multiply perforated domain.

Let Ω be a bounded open set in $\mathbb{R}^N$. We consider functions which depend on one macroscopic variable and n microscopic variables.

Let $\{a_1(\varepsilon)\}, \{a_2(\varepsilon)\}, \ldots, \{a_n(\varepsilon)\}$ be n sequences such that

$$\lim_{\varepsilon\to 0} a_i(\varepsilon) = 0 \text{ for } i = 1, 2, \ldots n \tag{4.1}$$

and such that $\exists m > 0$ with

$$\lim_{\varepsilon\to 0} \frac{1}{a_i(\varepsilon)} \left[\frac{a_{i+1}(\varepsilon)}{a_i(\varepsilon)} \right] = 0 \text{ for } i = 1, 2, \ldots n-1. \tag{4.2}$$

If (4.2) is satisfied, then we say that the scales $a_i(\varepsilon)$ are well-separated.

Example. Let $0 < k_1 < k_2 < \cdots < k_n$ be n numbers. Then we may take $a_i(\varepsilon) = \varepsilon^{k_i}$.

DEFINITION 4.1

For any Y_k-periodic function (for all $k = 1, 2, \ldots, n$) $\phi(x, y_1, \ldots, y_n)$, the oscillating function $[\phi]_\varepsilon$ is defined by

$$[\phi]_\varepsilon(x) = \phi\left(x, \frac{x}{a_1(\varepsilon)}, \cdots \frac{x}{a_n(\varepsilon)}\right).$$ □

DEFINITION 4.2

A sequence $u_\varepsilon \in L^2(\Omega)$ is said to $(n+1)$ -scale converge to a function $u \in L^2(\Omega \times Y_1 \times \cdots \times Y_n)$ if

$$\int_\Omega u_\varepsilon [\phi]_\varepsilon(x) \mathrm{d}x \to \int_\Omega \int_{Y_1} \cdots \int_{Y_n} (u\phi)(x, y_1, \ldots, y_n) \mathrm{d}y_n \ldots \mathrm{d}y_1 \mathrm{d}x \tag{4.3}$$

for all $\phi \in D(\Omega, C_\#^\infty(Y_1 \times \cdots \times Y_n))$, where $Y_i = [0,1]^N$ for $i = 1, 2, \ldots, n$. We write $u_\varepsilon \xrightarrow{(n+1)-s} u(x, y_1, \ldots, y_n)$. □

The following theorem justifies the definition of $(n+1)$ -scale convergence.

Theorem 4.1. (*Compactness*)(cf. [2]) *Every bounded sequence in $L^2(\Omega)$ has an $(n+1)$ -scale convergent subsequence.* □

Remark 4.1. (1) If $u(x, y_1, \ldots, y_n)$ in smooth, then $[u]_\varepsilon(x) \xrightarrow{(n+1)-s} u(x, y_1, \ldots, y_n)$.
(2) If $u_\varepsilon \to u$ in $L^2(\Omega)$ strongly, then $u_\varepsilon \xrightarrow{(n+1)-s} u(x)$.
(3) If $u_\varepsilon \xrightarrow{(n+1)-s} u(x, y_1, \ldots, y_n)$, then $u_\varepsilon \rightharpoonup \int_{Y_1} \cdot\cdot \int_{Y_n} u(x, y_1, \ldots, y_n) \mathrm{d}y_n .. \mathrm{d}y_1$ weakly in $L^2(\Omega)$.
(4) Thus any $(n+1)$-scale convergent sequence $\{u_\varepsilon\}$ is bounded in $L^2(\Omega)$. This follows from the previous remark and the uniform boundedness principle.
(5) If u_ε has the expansion

$$u_\varepsilon(x) = [u]_\varepsilon(x) + \sum_{i=1}^{n} a_i(\varepsilon)[u_i]_\varepsilon(x) + \cdots$$

where the functions $u, u_1, \ldots, u_n$ are assumed to be smooth, then

$$u_\varepsilon \xrightarrow{(n+1)-s} u(x, y_1, .., y_n).$$

Hence the $(n+1)$ -scale limit of a sequence determines the first term in the asymptotic expansion of $u_\varepsilon(x)$. □

DEFINITION 4.3

(a) A measurable function $u(x, y_1, .., y_n)$ is said to be admissible if

$$\int_\Omega ([u]_\varepsilon(x))^2 \mathrm{d}x \to \int_\Omega \int_{Y_1} \cdot\cdot \int_{Y_n} u(x, y_1, .., y_n)^2 \mathrm{d}y_n .. \mathrm{d}y_1 \mathrm{d}x. \tag{4.4}$$

(b) A sequence $\{u_\varepsilon\}$ which $(n+1)$-scale converges to an $u(x, y_1, .., y_n)$ is said to be admissible if

$$\int_\Omega (u_\varepsilon)^2 \mathrm{d}x \to \int_\Omega \int_{Y_1} \cdot\cdot \int_{Y_n} u(x, y_1, .., y_n)^2 \mathrm{d}y_n .. \mathrm{d}y_1 \mathrm{d}x. \tag{4.5}$$

□

Remark 4.2. (1) The space $L^\infty(\Omega, C_\#(Y_1 \times \cdots \times Y_n))$ is an example of a space of admissible functions. In general, continuity in n of the variables is sufficient in addition to the appropriate measurability properties (cf. [2]).
(2) If $u \in L^\infty(\Omega, C_\#(Y_1 \times \cdots \times Y_n))$, then $[u]_\varepsilon(x) \xrightarrow{(n+1)-s} u(x, y_1, .., y_n)$. □

The following theorem is in a certain sense, a theorem on strong $(n+1)$ -scale convergence. This is made precise as follows.

Theorem 4.2. [2] *Suppose that* $u_\varepsilon \xrightarrow{(n+1)-s} u(x, y_1, .., y_n)$ *and* $\{u_\varepsilon\}$ *is an admissible sequence. Also suppose that* $\{v_\varepsilon\}$ *is another sequence which* $(n+1)$*-scale converges to* $v(x, y_1, .., y_n)$. *Then,*

$$u_\varepsilon v_\varepsilon \to \int_{Y_1} \cdot\cdot \int_{Y_n} (uv)(x, y_1, .., y_n) \mathrm{d}y_n .. \mathrm{d}y_1 \ \text{in } D'(\Omega). \tag{4.6}$$

Also,

$$\int_\Omega u_\varepsilon v_\varepsilon \mathrm{d}x \to \int_\Omega \int_{Y_1} \cdot\cdot \int_{Y_n} (uv)(x, y_1, .., y_n) \mathrm{d}y_n .. \mathrm{d}y_1 \mathrm{d}x. \tag{4.7}$$

Further, if the $(n+1)$ *-scale limit* $u(x, y_1, .., y_n)$ *is admissible and*

$$[u]_\varepsilon(x) \xrightarrow{(n+1)-s} u(x, y_1, .., y_n),$$

then

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - [u]_\varepsilon(x)\|_{2,\Omega} = 0. \tag{4.8}$$

□

Remark 4.3. (a) In Theorem 4.2, the conclusion (4.7) is new though the proof of (4.7) is similar to that of (4.6).
(b) In fact, (4.6) can be improved to weak convergence in $L^2(\Omega)$ if $u_\varepsilon v_\varepsilon \in L^2(\Omega)$. This happens, for instance, if $u_\varepsilon = \chi_\varepsilon \doteq \prod_{i=1}^n \chi_i(x/a_i(\varepsilon))$ where $\chi_i(y_i)$ are the characteristic functions of $Y_i^* \subseteq Y_i(Y_i^*$ are certain subsets of the unit cell when we look at certain problems over a perforated domain). □

COROLLARY 4.1

Let $u_\varepsilon \xrightarrow{(n+1)-s} u(x, y_1, \ldots, y_n)$ *and* $\phi \in L^\infty(\Omega, C_\#(Y_1 \times \cdots \times Y_n))$. *Then*

$$u_\varepsilon[\phi]_\varepsilon \xrightarrow{(n+1)-s} (u\phi)(x, y_1, \ldots, y_n). \tag{4.9}$$

Proof. The proof is similar to the proof appearing in the 2-scale case (cf. Theorem 2.3). □

COROLLARY 4.2

Let $\{u_\varepsilon\}$ *be a sequence which* $(n+1)$*-scale converges to* $u(x, y_1, \ldots, y_n)$. *Assume that* $\{u_\varepsilon\}$ *is an admissible sequence. Let* $\phi \in L^\infty(\Omega, C_\#(Y_1 \times \cdots \times Y_n))$. *Let* w_ε *be another sequence which* $(n+1)$ *-scale converges to* $w(x, y_1, \ldots, y_n)$. *Then,*

$$\int_\Omega u_\varepsilon[\phi]_\varepsilon w_\varepsilon \mathrm{d}x \to \int_\Omega \int_{Y_1} \cdots \int_{Y_n} (u\phi w)(x, y_1, \ldots, y_n)\mathrm{d}y_n \ldots \mathrm{d}y_1\, \mathrm{d}x. \tag{4.10}$$

Proof. Apply Corollary 4.1 to ϕ, w_ε and take ϕw_ε for v_ε in Theorem 4.2. □

Theorem 4.3. [2] *Let* $\{u_\varepsilon\}$ *be a sequence in* $H^1(\Omega)$ *which converges weakly to* u in $H^1(\Omega)$. *Then for a subsequence (which we continue to index by* ε*) and functions* $u_k(x, y_1, \ldots, y_k) \in L^2(\Omega \times Y_1 \times \cdots \times Y_{k-1}, H^1_\#(Y_k^*)\backslash\mathbb{R})$, $k = 1, 2, \ldots, n$ *we have,*

$$u_\varepsilon \xrightarrow{(n+1)-s} u(x) \text{ and, } \nabla u_\varepsilon \xrightarrow{(n+1)-s} \nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k. \tag{4.11}$$

□

5. Convergence of quadratic functionals

We now apply the results of the previous section to obtain the limit of $J_\varepsilon(u_\varepsilon)$ where J_ε is a quadratic functional and u_ε is the solution to a Neumann problem in a multiscale perforated domain. As before we assume that the scales $a_1(\varepsilon), a_2(\varepsilon), \ldots, a_n(\varepsilon)$ are well separated. We now define a multi-scale periodically perforated domain Ω_ε for a fixed $\varepsilon > 0$.

Let $T_i (i = 1, 2, \ldots, n)$ be open subsets of $Y = [0, 1]^N$ with smooth boundary. Let $Y_i^* = Y_i \backslash T_i$. Define

$$S^\varepsilon = \bigcup_{\mathbf{k}\in\mathbb{Z}^N} \bigcup_{i=1}^n a_i(\varepsilon)(\mathbf{k} + T_i)$$

and set $T^\varepsilon = \Omega \cap S^\varepsilon$.

Define Ω_ε to be $\Omega\setminus\overline{T^\varepsilon}$. We assume Ω_ε to be connected. The boundary of Ω_ε consists of two parts; the union of boundaries of holes entirely contained in Ω viz., $\partial_{\text{int}}\Omega_\varepsilon = \bigcup_{\mathbf{k}\in\mathbb{Z}^N}\partial B(\mathbf{k})$ where,

$$B(\mathbf{k}) = \bigcup_{i=1}^{n} \{a_i(\varepsilon)(\mathbf{k} + T^i)\} \text{ and } \overline{B(\mathbf{k})}\subset\Omega$$

and the exterior boundary, $\partial_{\text{ext}}\Omega_\varepsilon = \partial\Omega_\varepsilon\setminus\partial_{\text{int}}\Omega_\varepsilon$. Consider the Neumann problem,

$$\begin{cases} -\operatorname{div}(A_\varepsilon\nabla u_\varepsilon) = f & \text{in}\,\Omega_\varepsilon \\ A_\varepsilon\nabla u_\varepsilon\cdot n_\varepsilon = 0 & \text{on } \partial_{\text{int}}\Omega_\varepsilon \\ u = 0 & \text{on } \partial_{\text{ext}}\Omega_\varepsilon \end{cases}, \tag{5.1}$$

where $A_\varepsilon(x) = (([a_{ij}]_\varepsilon(x)))$ and we have the following assumptions on $A = ((a_{ij}))$.

(1) $A\in M(\alpha,\beta,\Omega\times Y_1\times\cdots\times Y_n)$.
(2) $a_{ij}\in L^\infty(\Omega, C_\#(Y_1\times\cdots\times Y_n))$.

It is known that (5.1) has a unique solution u_ε and that the sequence $\{\tilde{u}_\varepsilon\}$ is bounded in $L^2(\Omega)$ independently of ε. Assume that $\tilde{u}_\varepsilon\to\lambda u$ in $L^2(\Omega)$ where $\lambda = \prod_{i=1}^n |Y_i^*|$ and $u\in L^2(\Omega)$. Then it is known that u solves the homogenized problem given by the following theorem.

Theorem 5.1. [2] *Let u_ε be the solution of* (5.1). *Then,*

$$\tilde{u}_\varepsilon \xrightarrow{(n+1)-s} u(x)\chi(y_1,\dots,y_n)$$

$$\widetilde{\nabla u_\varepsilon} \xrightarrow{(n+1)-s} \left(\nabla_x u + \sum_{k=1}^{n}\nabla_{y_k}u_k\right)\chi(y_1,\dots,y_n), \tag{5.2}$$

where $\chi(y_1,\dots,y_n) \doteq \prod_{i=1}^n \chi_i(y_i)$ and χ_i is the characteristic function of the set Y_i^*. Also, $(u,u_1,\dots,u_n)$ is the unique solution in

$$V = H_0^1(\Omega)\times\prod_{i=1}^{n}(L^2(\Omega\times Y_1\times\cdots\times Y_{i-1}, H_\#^1(Y_i^*))$$

of the $(n+1)$-scale homogenized problem:

$$\begin{cases} -\operatorname{div}_{yn}(A(\nabla_x u + \Sigma_{k=1}^n\nabla_{y_k}u_k)) = 0 & \text{in } Y_n^* \\ A(\nabla_x u + \Sigma_{k=1}^n\nabla_{y_k}u_k)\cdot n = 0 & \text{on } \partial T_n \\ \int_{Y_n}\chi_n(yn)u_n\,dy_n = 0 & \\ -\operatorname{div}_{y_j}(\int_{Y_{j+1}}\cdots\int_{Y_n}\prod_{j+1}^n\chi_k(y_k)A(\nabla_x u + \Sigma_{k=1}^n\nabla_{y_k}u_k)\,dy_n\cdots dy_{j+1}) = 0 & \text{in } Y_j^* \\ (\int_{Y_{j+1}}\cdots\int_{Y_n}\prod_{j+1}^n\chi_k(y_k)A(\nabla_x u + \Sigma_{k=1}^n\nabla_{y_k}u_k)\,dy_n\cdots dy_{j+1})\cdot n = 0 & \text{on } \partial T_j \\ \int_{Y_j}\chi_j(y_j)u_j\,dy_j = 0 & \\ \text{for } j = 1,2,\dots,n-1 \text{ and finally,} & \\ -\operatorname{div}_x\int_{Y_1}\cdots\int_{Y_n}\prod_1^n\chi_k(y_k)A(\nabla_x u + \Sigma_{k=1}^n\nabla_{y_k}u_k)\,dy_n\cdots dy_1 = \lambda f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

□

COROLLARY 5.1

The function u is also the unique solution of the homogenized problem

$$\begin{cases} -\operatorname{div}(A^0\nabla u) = \lambda f \text{ in } \Omega \\ \qquad u = 0 \text{ on } \partial\Omega \end{cases} \tag{5.4}$$

where A^0 is obtained from the iterative formulae,

$$\begin{cases} A^n = A \\ A^{j-i}\xi = \int_{Y_j}\chi_k(y_k)A^j(\xi + \nabla w_j^\xi)\mathrm{d}y_j \end{cases} \tag{5.5}$$

for $1 \leqslant j \leqslant n$ and, for $\xi \in \mathbb{R}^N$, w_j^ξ is the unique solution of the cell equation:

$$\begin{cases} -\operatorname{div}_{y_j}(A^j(\xi + \nabla w_j^\xi)) = 0 \quad \text{in } Y_j^* \\ A^j(\xi + \nabla w_j^\xi)\cdot n = 0 \quad \text{on } \partial T_j \\ \int_{Y_j}\chi_j(y_j)w_j^\xi(x, y_1, \ldots, y_j)\mathrm{d}y_j = 0 \end{cases} \tag{5.6}$$

where $w_j^\xi \in L^2(\Omega \times Y_1 \times \cdots \times Y_{j-1}; H^1(Y_j^*))$. □

We now consider cost-functionals

$$J_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \mathrm{d}x$$

where $B_\varepsilon \doteq (([b_{ij}]_\varepsilon(x)))$ and $b_{ij} \in L^\infty(\Omega, C_\#(Y_1 \times \cdots \times Y_n))$. In all that follows let u_ε be the solution of (5.1) and $(u, u_1, \ldots, u_n)$ be the solution of the homogenized problem (5.3). Let us assume that $u_k \in L^2(\Omega, C_\#^1(Y_1 \times \cdots \times Y_k))$ so that u_k may be used as test functions in multi-scale convergence. Then we show that,

$$\begin{aligned} \lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) = \int_\Omega\int_{Y_1}\cdots\int_{Y_n} \chi(\mathbf{y})B\left(\nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k\right) \\ \times \left(\nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k\right)\mathrm{d}y_n..\mathrm{d}y_1\mathrm{d}x, \end{aligned} \tag{5.7}$$

where $\mathbf{y} \equiv (y_1, \ldots, y_n)$ with $y_i \in Y_i$. The proof of the convergence of cost-functionals i.e. the result (5.7) goes along the same lines as in the 2-scale case. First we require the following lemma.

Lemma 5.1. *With $(u, u_1, \ldots, u_n)$ as above and assumed to be regular, the sequence*

$$\left\{\chi_\varepsilon\left(\frac{\partial u}{\partial x_i} + \sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}}\left(x, \frac{x}{a_1(\varepsilon)}, \ldots, \frac{x}{a_k(\varepsilon)}\right)\right)\right\}$$

is admissible for $i = 1, 2, \ldots, N$.

Proof. First we note that,

$$\chi_\varepsilon\left(\frac{\partial u}{\partial x_i} + \sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}}\left(x, \frac{x}{a_1(\varepsilon)}, \ldots, \frac{x}{a_k(\varepsilon)}\right)\right) \xrightarrow{(n+1)-s} \chi\left(\frac{\partial u}{\partial x_i} + \sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}}\right)$$

and

$$\begin{aligned} \int_\Omega \left(\frac{\partial u}{\partial x_i}\chi_\varepsilon + \sum_{k=1}^n \left[\frac{\partial u_k}{\partial y_{ki}}\right]_\varepsilon(x)\chi_\varepsilon\right)^2 \mathrm{d}x = \int_\Omega \left(\frac{\partial u}{\partial x_i}\right)^2 \chi_\varepsilon + 2\frac{\partial u}{\partial x_i}\chi_\varepsilon \sum_{k=1}^n \left[\frac{\partial u_k}{\partial y_{ki}}\right]_\varepsilon(x) \\ + \left(\sum_{k=1}^n \left[\frac{\partial u_k}{\partial y_{ki}}\right]_\varepsilon(x)\right)^2 \mathrm{d}x, \end{aligned}$$

where as usual

$$\left[\frac{\partial u_k}{\partial y_{ki}}\right]_\varepsilon (x) \doteq \frac{\partial u_k}{\partial y_{ki}}\left(x, \frac{x}{a_1(\varepsilon)}, \ldots, \frac{x}{a_k(\varepsilon)}\right).$$

Now,

$$\chi_\varepsilon \to \int_{Y_1} \cdots \int_{Y_n} \chi(y_1, \ldots, y_n) \mathrm{d}y_n \ldots \mathrm{d}y_n \ \text{ in } L^\infty(\Omega) \text{ weak } *$$

and,

$$\left(\frac{\partial u}{\partial x_i}\right)^2 \in L^1(\Omega).$$

Therefore,

$$\int_\Omega \left(\frac{\partial u}{\partial x_i}\right)^2 \chi_\varepsilon \mathrm{d}x \to \int_\Omega \int_{Y_1} \cdots \int_{Y_n} \left(\frac{\partial u}{\partial x_i}\right)^2 \chi \mathrm{d}y_n \ldots \mathrm{d}y_1 \mathrm{d}x.$$

Next,

$$\frac{\partial u}{\partial x_i}\chi_\varepsilon \xrightarrow{(n+1)-s} \frac{\partial u}{\partial x_i}\chi(y_1, \ldots, y_n)$$

and $\{\Sigma_{k=1}^n [\partial u_k/\partial y_{ki}]_\varepsilon (x)\}$ is an admissible sequence since we have assumed the u_ks to be regular. Therefore by Theorem 4.2,

$$\int_\Omega \frac{\partial u}{\partial x_i}\chi_\varepsilon \left(\sum_{k=1}^n \left[\frac{\partial u_k}{\partial y_{ki}}\right]_\varepsilon (x)\right) \mathrm{d}x \to \int_\Omega \int_{Y_1} \cdots \int_{Y_n} \chi \frac{\partial u}{\partial x_i}\left(\sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}}\right) \mathrm{d}y_n \ldots \mathrm{d}y_1 \mathrm{d}x.$$

Finally,

$$\chi_\varepsilon \left(\sum_{k=1}^n \left[\frac{\partial u_k}{\partial y_{ki}}\right]_\varepsilon (x)\right) \xrightarrow{(n+1)-s} \chi\left(\sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}}\right)$$

by Corollary 4.1. Therefore,

$$\int_\Omega \left(\sum_{k=1}^n \left[\frac{\partial u_k}{\partial y_{ki}}\right]_\varepsilon (x)\right)^2 \chi_\varepsilon \mathrm{d}x \to \int_\Omega \int_{Y_1} \cdots \int_{Y_n} \chi\left(\sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}}\right)^2 \mathrm{d}y_n \ldots \mathrm{d}y_1 \mathrm{d}x$$

by Corollary 4.2. Hence,

$$\int_\Omega \chi_\varepsilon \left(\frac{\partial u}{\partial x_i} + \sum_{k=1}^n \left[\frac{\partial u_k}{\partial y_{ki}}\right]_\varepsilon (x)\right)^2 \mathrm{d}x \to \int_\Omega \int_{Y_1} \cdots \int_{Y_n} \chi \left(\frac{\partial u}{\partial x_i} + \sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}}\right)^2 \ \mathrm{d}y_n \ldots \mathrm{d}y_1 \mathrm{d}x$$

which proves the admissibility of the said sequence. □

Then we obtain, as in the 2-scale case, the following corrector result. The proof being similar, we omit it.

Theorem 5.2. *If u_ε is the solution of the Neumann problem* (5.1) *and we assume further that $(u, u_1, \ldots, u_n)$ are sufficiently regular then,*

$$\lim_{\varepsilon \to 0} \left\| \widetilde{\frac{\partial u_\varepsilon}{\partial x_i}} - \frac{\partial u}{\partial x_i}\chi_\varepsilon - \sum_{k=1}^n \left[\frac{\partial u_k}{\partial y_{ki}}\right]_\varepsilon (x)\chi_\varepsilon \right\|_{2,\Omega} = 0 \ \textit{for} \ i = 1, 2, \ldots, N. \qquad \square$$

We now show that $\{\widetilde{\partial u_\varepsilon/\partial x_i}\}$ is an admissible sequence. Before that we prove the following lemma.

Lemma 5.2. *Let* $a_\varepsilon \xrightarrow{(n+1)-s} a(x, y_1, \ldots, y_n)$ *and assume that* $\{a_\varepsilon\}$ *is an admissible sequence. Let* $\{b_\varepsilon\}$ *be a sequence of functions in* $L^2(\Omega)$ *such that*

$$\lim_{\varepsilon \to 0} \| a_\varepsilon - b_\varepsilon \|_{L^2(\Omega)} = 0. \tag{5.8}$$

Then $\{b_\varepsilon\}$ *is also admissible.*

Proof. Note that we can write, $b_\varepsilon = a_\varepsilon + t_\varepsilon$ for a sequence $t_\varepsilon \in L^2(\Omega)$ such that $t_\varepsilon \to 0$ strongly in $L^2(\Omega)$. We have that $t_\varepsilon \xrightarrow{(n+1)-s} 0$ (since $t_\varepsilon \to 0$ strongly in $L^2(\Omega)$). Therefore, $b_\varepsilon \xrightarrow{(n+1)-s} a(x, y_1, \ldots, y_n)$ and,

$$\int_\Omega (b_\varepsilon)^2 \mathrm{d}x = \int_\Omega (a_\varepsilon)^2 \mathrm{d}x + 2\int_\Omega a_\varepsilon t_\varepsilon \mathrm{d}x + \int_\Omega (t_\varepsilon)^2 \mathrm{d}x. \tag{5.9}$$

Hence,

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_\Omega (b_\varepsilon)^2 \mathrm{d}x &= \lim_{\varepsilon \to 0} \int_\Omega (a_\varepsilon)^2 \mathrm{d}x \\ &= \int_\Omega \int_{Y_1} \cdots \int_{Y_n} a(x, y_1, \ldots, y_n)^2 \mathrm{d}x\end{aligned}$$

(since the second and third terms in right hand side of (5.9) tend to zero by (5.8)). Hence, $\{b_\varepsilon\}$ is an admissible sequence. □

Let $\{u_\varepsilon\}$, $u, u_1, \ldots, u_n$ and the matrix B be as before. Then we have the following theorem.

Theorem 5.3. *Assuming that* $u, u_1, \ldots, u_n$ *are sufficiently smooth we have,*

$$\lim_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon) = \int_\Omega \int_{Y_1} \cdots \int_{Y_n} B\left(\nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k\right) \cdot \left(\nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k\right) \mathrm{d}y_n..\mathrm{d}y_1 \mathrm{d}x.$$

Proof. We have,

$$J_\varepsilon(u_\varepsilon) = \int_\Omega [b_{ij}]_\varepsilon \frac{\widetilde{\partial u_\varepsilon}}{\partial x_j} \frac{\widetilde{\partial u_\varepsilon}}{\partial x_i} \mathrm{d}x.$$

It follows from Lemma 5.1, Theorem 5.2 and Lemma 5.2 that $\{\widetilde{\partial u_\varepsilon/\partial x_i}\}$ is an admissible sequence. Further

$$\frac{\widetilde{\partial u_\varepsilon}}{\partial x_i} \xrightarrow{(n+1)-s} \chi\left(\frac{\partial u}{\partial x_i} + \sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}}\right) \qquad \forall i = 1, 2, \ldots, N \text{ and,}$$

$$b_{ij} \in L^\infty(\Omega, C_\#(Y_1 \times \cdots \times Y_n)).$$

So, by Corollary 4.2,

$$\lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0}\int_\Omega B_\varepsilon \widetilde{\nabla u_\varepsilon}\cdot\widetilde{\nabla u_\varepsilon}\,\mathrm{d}x$$

$$= \int_\Omega\int_{Y_1}\cdots\int_{Y_n} \chi B\left(\nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k\right)\cdot\left(\nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k\right)\mathrm{d}y_n..\mathrm{d}y_1\,\mathrm{d}x.$$

□

Remark 5.2. By an iterative formula we can write the limit of $J_\varepsilon(u_\varepsilon)$ as $\int_\Omega B^0\nabla u\cdot\nabla u\,\mathrm{d}x$ where B^0 is given by the iterative formulae,

$$B^n = B,$$

$$(B^{k-1})_{ij} = (H_0 \text{ limit of } B^k)_{ij} + \int_{Y_k}\chi_k(y_k)B^k\nabla_{yk}(w_k^i - \eta_k^i)\nabla_{y_j}(w_k^j - \eta_k^j)\mathrm{d}y_k$$

for all $1\leqslant i,j\leqslant N$ and for $k = 1,2,\ldots,n$; where, η_k^i is the unique solution of the ith cell equation for B^k viz.,

$$-\mathrm{div}_{yk}(B^k(e^i + \nabla_{y_k}\eta_k^i)) = 0 \quad \text{in } Y_k^*$$
$$B^k(e^i + \nabla_{y_k}\eta_k^i)\cdot n = 0 \quad \text{on } \partial T_k$$
$$\int_{Y_k}\chi_k(y_k)\eta_k^i\,\mathrm{d}y_k = 0.$$

and w_k^i are obtained from (5.6) by taking $\xi = e^i$.

References

[1] Allaire G, Homogenization and two-scale convergence, *SIAM J. Math. Anal.* **23** (1992) 1482–1518

[2] Allaire G and Briane M, Multiscale convergence and reiterated homogenization, *Publ. du Laboratoire d'Analyse Numérique*, R94019 (1994)

[3] Allaire G and Murat M, Homogenization of the Neumann problem with nonisolated holes, *Asymptotic Anal.* **7** (1993) 81–95

[4] Kesavan S and Saint Jean Paulin J, Homogenization of an optimal control problem, *SIAM J. Control. Optim.* **35** (1997) 1557–1573

[5] Kesavan S and Saint Jean Paulin J, *Optimal control on perforated domains*, to appear (1997)

[6] Kesavan S and Vanninathan M, L'homogénéisation d'un probléme de contrôle optimal, *C.R.A.S Série A, t.* **285** (1977) 441–444

[7] Lions J L, *Optimal Control of Systems Governed by Partial Differential Equations* (Springer-Verlag) (1971)

[8] Nguetseng G, A general convergence result of a functional related to the theory of homogenization, *SIAM J. Math. Anal.* **20** (1989) 608–623

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 2, June 1998, pp. 209–215.

Variational and reciprocal principles in thermoelasticity without energy dissipation

D S CHANDRASEKHARAIAH

Department of Mathematics, Bangalore University, Central College Campus, Bangalore 560 001, India

MS received 19 January 1998

Abstract. In the context of the linear theory of thermoelasticity without energy dissipation for homogeneous and isotropic materials, variational principles of Biot- and Hamilton-types and a reciprocal principle of Betti-Rayleigh-type are presented.

Keywords. Variational principle; Hamilton's principle; reciprocal principle.

1. Introduction

In 1993, Green and Naghdi [6] formulated a generalized thermoelasticity theory for homogeneous and isotropic materials by including the so-called 'thermal-displacement gradient' among the independent constitutive variables. An important feature of the linearized version of this theory, which is not present in the conventional coupled thermoelasticity theory [1] and other generalized thermoelasticity theories [2], is that it does not accomodate dissipation of thermal energy. In the context of this linearized theory of Green and Naghdi (which we will henceforth refer to as the GN theory), theorems on uniqueness of solutions have been proved in [3, 4].

In this article, we present two variational principles and a reciprocal principle in the context of the GN theory. One of the variational principles is of the Biot-type [1] and the other variational principle is of the Hamilton-type [8]. The reciprocal principle is of the Betti-Rayleigh type [7].

It may be mentioned that the variational and reciprocal principles are not only of mathematical interest but are of practical utility also [5, 7, 8].

2. Governing equations

The governing equations of the GN theory are [6, 3]

$$\mathbf{T} = \lambda(\operatorname{div}\mathbf{u})\mathbf{I} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) - \gamma\theta\mathbf{I}, \tag{1}$$

$$\rho\eta = c\frac{\theta}{\theta_0} + \gamma\operatorname{div}\mathbf{u}, \tag{2}$$

$$\dot{\mathbf{p}} = -\frac{\kappa^*}{\theta_0}\nabla\theta, \tag{3}$$

$$\operatorname{div}\mathbf{T} + \rho\mathbf{b} = \rho\ddot{\mathbf{u}}, \tag{4}$$

$$\rho\dot{\eta} = \frac{\rho}{\theta_0}r - \operatorname{div}\mathbf{p}. \tag{5}$$

In the above equations and those that follow, the symbols and the notation are as explained in [3], unless stated to the contrary.

Elimination of $\mathbf{T}$, $\mathbf{p}$ and η from equations (1)–(5) yields the following field equations expressed completely in terms of $\mathbf{u}$ and θ [6]:

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla \operatorname{div}\mathbf{u} - \gamma\nabla\theta + \rho\mathbf{b} = \rho\ddot{\mathbf{u}}, \tag{6}$$

$$\kappa^*\nabla^2\theta + \rho\dot{r} = c\ddot{\theta} + \gamma\theta_0 \operatorname{div}\ddot{\mathbf{u}}. \tag{7}$$

To the above system of field equations, we adjoin the initial conditions

$$\mathbf{u} = \mathbf{0},\ \dot{\mathbf{u}} = \mathbf{0},\ \theta = 0,\ \dot{\theta} = 0 \ \text{ in } B \text{ at } t = 0 \tag{8}$$

and the boundary conditions

$$\mathbf{u} = \mathbf{u}^* \ \text{ on } \partial B_1, \mathbf{Tn} = t^* \ \text{ on } \partial B_1^c \ \text{ for } t \in [0, \infty), \tag{9}$$

$$\theta = \theta^* \ \text{ on } \partial B_2, \nabla\theta\cdot\mathbf{n} = q^* \ \text{ on } \partial B_2^c \ \text{ for } t \in [0, \infty). \tag{10}$$

Here, ∂B_1 and ∂B_2 are arbitrary parts of the boundary surface ∂B of a regular region B with ∂B_1^c and ∂B_2^c as their respective complements in ∂B, and $\mathbf{n}$ is the unit outward normal to ∂B. Also, $\mathbf{u}^*$, $\mathbf{t}^*$, θ^* and q^* are prescribed functions in their respective domains.

The uniqueness of solution of the initial mixed boundary value problem associated with the field equations (6) and (7) and the initial and boundary conditions (8)–(10) has been established in [4]. (Indeed, the uniqueness of solution proved in [4] holds in the case where the initial conditions are not necessarily homogeneous).

3. Variational principle-I

We now present the variational principle of the Biot-type [1] in the form of the following theorem.

Theorem 1. *Suppose that* $\mathbf{b}$ *and* r *are prescribed in* B *for* $t \geqslant 0$ *and that the initial and boundary conditions* (8)–(10) *hold. Let*

$$V = \frac{1}{2}\int_B \left\{\lambda(\operatorname{div}\mathbf{u})^2 + \mu\nabla\mathbf{u}\cdot(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) + \frac{c}{\theta_0}\theta^2\right\} \mathrm{d}B, \tag{11}$$

$$U = \frac{1}{2}\int_B \rho(\tau^2\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}B, \tag{12}$$

$$P = \int_B \rho\mathbf{b}\cdot\mathbf{u}\,\mathrm{d}B, \tag{13}$$

$$D = \frac{1}{2}\frac{\theta_0}{\kappa^*}\int_B \tau^2(\mathbf{s}\cdot\mathbf{s})\mathrm{d}B, \tag{14}$$

$$G = \int_{\partial B_1^c} \mathbf{t}^*\cdot\mathbf{u}\,\mathrm{d}A, \tag{15}$$

$$H = \int_{\partial B_2} \theta^*\mathbf{s}\cdot\mathbf{n}\,\mathrm{d}A, \tag{16}$$

where $\tau = \partial/\partial t$ *and* $\mathbf{s}$ *is the entropy flow vector* [1] *defined by*

$$\dot{\mathbf{s}} = \mathbf{p}. \tag{17}$$

Then, for arbitrary variations $\delta\mathbf{u}$ *and* $\delta\mathbf{s}$ *in* $\mathbf{u}$ *and* $\mathbf{s}$ *respectively, which are consistent with the boundary conditions* (9) and (10) *and other kinematic constraints, the variational equation*

$$\delta(V+U-P+D-G+H)=0, \tag{18}$$

wherein the operator τ *is treated as a constant, yields the field equations* (6) *and* (7).

Proof. We first note, with the aid of equations (17), (5) and (2), that

$$\operatorname{div}\dot{\mathbf{s}}=\frac{\rho}{\theta_0}r-\left(c\frac{\dot{\theta}}{\theta_0}+\gamma\operatorname{div}\dot{\mathbf{u}}\right). \tag{19}$$

Bearing in mind that r is prescribed in B and that the initial conditions are homogeneous, we find from (19) that

$$\operatorname{div}(\delta\mathbf{s})=-\frac{c}{\theta_0}\delta\theta-\gamma\operatorname{div}(\delta\mathbf{u}). \tag{20}$$

Now, let us compute the variations δV, δU, δP, δD, δG and δH in V, U, P, D, G and H respectively.

First, from equation (11) we obtain

$$\delta V=\int_B\left\{\lambda(\operatorname{div}\mathbf{u})\delta(\operatorname{div}\mathbf{u})+\mu(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}})\cdot\delta(\nabla\mathbf{u})+\frac{c}{\theta_0}\theta\delta\theta\right\}\mathrm{d}B \tag{21}$$

which upon using equation (1) and the divergence theorem becomes

$$\delta V=\int_{\partial B}(\mathbf{T}+\gamma\theta\mathbf{I})\delta\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}A$$
$$-\int_B\left[\{\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla\operatorname{div}\mathbf{u}\}\cdot\delta\mathbf{u}-\frac{c}{\theta_0}\theta\delta\theta\right]\mathrm{d}B. \tag{22}$$

Next, bearing in mind that $\mathbf{b}$ is prescribed in B and that the operator τ is to be treated as constant in the computations of δU and δD, we find from equations (12)–(14) that

$$\delta(U-P)=\int_B\rho(\ddot{\mathbf{u}}-\mathbf{b})\cdot\delta\mathbf{u}\,\mathrm{d}B, \tag{23}$$

$$\delta D=\int_B\frac{\theta_0}{\kappa^*}\ddot{\mathbf{s}}\cdot\delta\mathbf{s}\,\mathrm{d}B. \tag{24}$$

Lastly, in view of the mixed boundary conditions (9) and (10) and equations (3) and (17), we find from expressions (15) and (16) that

$$\delta(-G+H)=-\int_{\partial B}\{\mathbf{T}\delta\mathbf{u}-\theta\delta\mathbf{s}\}\cdot\mathbf{n}\,\mathrm{d}A. \tag{25}$$

With the aid of the divergence theorem, equation (20) and a standard vector identity, we find that

$$\int_{\partial B}\theta\delta\mathbf{s}\cdot\mathbf{n}\,\mathrm{d}A=\int_B\left\{\nabla\theta\cdot\delta\mathbf{s}+\gamma\nabla\theta\cdot\delta\mathbf{u}-\frac{c}{\theta_0}\theta\delta\theta\right\}\mathrm{d}B-\int_{\partial B}\gamma\theta\delta\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}A. \tag{26}$$

Adding expressions (22)–(25) and using (26), we obtain

$$\delta(V+U-P+D-G+H)=\int_B\left\{\frac{\theta_0}{\kappa^*}\ddot{\mathbf{s}}+\nabla\theta\right\}\cdot\delta\mathbf{s}\,\mathrm{d}B$$

$$+\int_B[\rho\ddot{\mathbf{u}}-\rho\mathbf{b}-\{\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla\operatorname{div}\mathbf{u}\}+\gamma\nabla\theta]\cdot\delta\mathbf{u}\,\mathrm{d}B. \tag{27}$$

Now, it is evident that the variational equation (18) yields the equation of motion (6) and the following equation:

$$\frac{\kappa^*}{\theta_0}\nabla\theta+\ddot{\mathbf{s}}=\mathbf{0}. \tag{28}$$

Taking the divergence of this equation and using equation (19) we obtain the heat transport equation (7). This completes the proof of the theorem.

4. Variational principle-II

We now establish a variational principle of the Hamilton-type [8] in the form of the following theorem.

Theorem 2. *Suppose that* $\mathbf{b}$ *and* r *are prescribed in* B *for* $t\geqslant 0$ *and that the initial and boundary conditions* (8)–(10) *hold. Also, let* t_1 *and* t_2 *be two arbitrary instants of time, and*

$$K=\frac{1}{2}\int_B\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\,\mathrm{d}B. \tag{29}$$

Then the field equations (6) *and* (7) *form a set of necessary and sufficient conditions for the variational equations*

$$\delta\int_{t_1}^{t_2}(K-V+G+P)\,\mathrm{d}t=0, \tag{30}$$

$$\delta\int_{t_1}^{t_2}\mathrm{d}t\left[\int_B\frac{1}{2}\kappa^*\nabla\theta\cdot\nabla\theta\,\mathrm{d}B-\int_{\partial B_2^c}\kappa^*q^*\theta\,\mathrm{d}A-\int_B\rho\tau(\eta\theta_0\dot{\theta}+r\theta)\,\mathrm{d}B\right]=0 \tag{31}$$

to hold for arbitrary variations $\delta\mathbf{u}$ *and* $\delta\theta$ *in* $\mathbf{u}$ *and* θ *respectively, which in addition to being compatible with the boundary conditions* (9) *and* (10) *and other kinematic constraints, satisfy the end conditions*

$$\delta\mathbf{u}=\mathbf{0},\ \delta\theta=0\ \text{ in } B \text{ for } t=t_1,t_2. \tag{32}$$

Further, in the evaluation of the variational equations (30) *and* (31), η *and* t *are held fixed and* τ *is treated as a constant.*

Proof. By using expression (29) and the first of the conditions (32), we obtain

$$\delta\int_{t_1}^{t_2}K\,\mathrm{d}t=-\int_{t_1}^{t_2}\mathrm{d}t\int_B\rho\ddot{\mathbf{u}}\cdot\delta\mathbf{u}. \tag{33}$$

With the aid of expressions (11), (1), (2), the divergence theorem and the boundary conditions (9), and bearing in mind that η is not submitted to variation, we obtain

$$\delta\int_{t_1}^{t_2} V\mathrm{d}t=\int_{t_1}^{t_2}\mathrm{d}t\bigg[\int_{\partial B_1^c}\mathbf{t}^*\cdot\delta\mathbf{u}\,\mathrm{d}A$$

$$-\int_B\{\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla\operatorname{div}\mathbf{u}-\gamma\nabla\theta\}\cdot\delta\mathbf{u}\,\mathrm{d}B\bigg]. \tag{34}$$

With the use of expressions (13), (15), (33) and (34), the variational equation (30) becomes

$$\int_{t_1}^{t_2}\mathrm{d}t\int_B[\{\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla\operatorname{div}\mathbf{u}-\gamma\nabla\theta+\rho\mathbf{b}-\rho\ddot{\mathbf{u}}\}\cdot\delta\mathbf{u}]\mathrm{d}B=0. \tag{35}$$

Obviously, this equation holds if and only if the equation of motion (6) is satisfied.

Next, by using the divergence theorem and the boundary conditions (10), we find that

$$\delta\int_{t_1}^{t_2}\mathrm{d}t\bigg[\int_B\frac{1}{2}\kappa^*\nabla\theta\cdot\nabla\theta\,\mathrm{d}B-\int_{\partial B_2^c}\kappa^*q^*\theta\,\mathrm{d}A\bigg]=-\int_{t_1}^{t_2}\mathrm{d}t\int_B\kappa^*(\nabla^2\theta)\delta\theta\,\mathrm{d}B. \tag{36}$$

With the aid of equation (36), the variational equation (31) reduces to

$$\int_{t_1}^{t_2}\mathrm{d}t\int_B\{\kappa^*\nabla^2\theta+\tau\rho(r-\dot{\eta}\theta_0)\}\delta\theta\,\mathrm{d}B=-\int_B\tau[\rho\eta\theta_0\delta\theta|_{t_2}^{t_1}]\mathrm{d}B. \tag{37}$$

Together with (32) and (2), equation (37) becomes

$$\int_{t_1}^{t_2}\mathrm{d}t\int_B\{\kappa^*\nabla^2\theta+\rho\dot{r}-c\ddot{\theta}-\gamma\theta_0\operatorname{div}\ddot{\mathbf{u}}\}\delta\theta\,\mathrm{d}B=0. \tag{38}$$

Obviously, this equation holds if and only if the heat transport equation (7) is satisfied. This completes the proof of the theorem.

5. Reciprocal principle

We now present the reciprocal principle of the Betti-Rayleigh type [7] in the form of the following theorem.

Theorem 3. *Consider two initial, mixed boundary value problems associated with the two systems*

$$\{(\mathbf{b}^{(\alpha)},r^{(\alpha)});(\mathbf{t}^{*(\alpha)},\mathbf{u}^{*(\alpha)},\theta^{*(\alpha)},q^{*(\alpha)});(\mathbf{u}^{(\alpha)},\mathbf{T}^{(\alpha)},\theta^{(\alpha)})\},\quad \alpha=1,2.$$

For $\alpha,\beta=1,2$, let

$$L_{\alpha\beta}=\int_B\rho\left\{\mathbf{b}^{(\alpha)}\circ\tilde{\mathbf{u}}^{(\beta)}-\frac{r^{(\alpha)}}{\theta_0}\circ\tilde{\theta}^{(\beta)}\right\}\mathrm{d}B-\int_{\partial B_1}\mathbf{u}^{*(\alpha)}\circ(\tilde{\mathbf{T}}^{(\beta)}\mathbf{n})\mathrm{d}A$$

$$+\int_{\partial B_1^c}\mathbf{t}^{*(\alpha)}\circ\tilde{\mathbf{u}}^{(\beta)}\mathrm{d}A+\frac{\kappa^*}{\theta_0}\left\{\int_{\partial B_2}\theta^{*(\alpha)}\circ(\nabla\theta^{(\beta)}\cdot\mathbf{n})\mathrm{d}A-\int_{\partial B_2^c}q^{*(\alpha)}\circ\theta^{(\beta)}\mathrm{d}A\right\}, \tag{39}$$

where

$$\mathbf{f}\circ\tilde{\mathbf{g}}=\int_0^t \mathbf{f}(\mathbf{x},t-v)\cdot\frac{\partial^2\mathbf{g}}{\partial t^2}(\mathbf{x},v)\mathrm{d}v, \tag{40}$$

$$f\circ g=\int_0^t f(\mathbf{x},t-v)g(\mathbf{x},v)\mathrm{d}v, \tag{41}$$

$$f\circ\hat{g}=\int_0^t f(\mathbf{x},t-v)\frac{\partial g}{\partial t}(\mathbf{x},v)\mathrm{d}v. \tag{42}$$

Then

$$L_{12}=L_{21}. \tag{43}$$

Proof. By hypothesis, the functions associated with the two problems considered are governed by the field equations (6) and (7) and the initial and boundary conditions (8)–(10). The Laplace transform versions of the field equations governing the two problems read as follows:

$$\mu\nabla^2\bar{\mathbf{u}}^{(\alpha)}+(\lambda+\mu)\nabla\operatorname{div}\bar{\mathbf{u}}^{(\alpha)}-\gamma\nabla\bar{\theta}^{(\alpha)}+\rho\bar{\mathbf{b}}^{(\alpha)}=\rho\xi^2\bar{\mathbf{u}}^{(\alpha)}, \tag{44}$$

$$\kappa^*\nabla^2\bar{\theta}^{(\alpha)}+\rho\xi\bar{r}^{(\alpha)}=c\xi^2\bar{\theta}^{(\alpha)}+\gamma\theta_0\xi^2\operatorname{div}\bar{\mathbf{u}}^{(\alpha)}. \tag{45}$$

Also, the Laplace transform versions of the boundary conditions for the problems read

$$\bar{\mathbf{u}}^{(\alpha)}=\bar{\mathbf{u}}^{*(\alpha)} \text{ on } \partial B_1,\ \bar{\mathbf{T}}^{(\alpha)}\mathbf{n}=\bar{\mathbf{t}}^{*(\alpha)} \text{ on } \partial B_1^c \quad \text{for } t\in[0,\infty), \tag{46}$$

$$\bar{\theta}^{(\alpha)}=\bar{\theta}^{*(\alpha)} \text{ on } \partial B_2,\ \nabla\bar{\theta}^{(\alpha)}\cdot\mathbf{n}=\bar{q}^{*(\alpha)} \text{ on } \partial B_2^c \quad \text{for } t\in[0,\infty). \tag{47}$$

In the above equations, an overbar denotes the Laplace transform of the corresponding function, ξ is the transform parameter, and $\alpha=1,2$.

With the use of eq. (44), the divergence theorem and the boundary conditions (46), we obtain

$$\begin{aligned}\int_B \rho(\bar{\mathbf{b}}^{(1)}\cdot\bar{\mathbf{u}}^{(2)}-\bar{\mathbf{b}}^{(2)}\cdot\bar{\mathbf{u}}^{(1)})\mathrm{d}B &= \int_B \gamma(\bar{\theta}^{(2)}\operatorname{div}\bar{\mathbf{u}}^{(1)}-\bar{\theta}^{(1)}\operatorname{div}\bar{\mathbf{u}}^{(2)})\mathrm{d}B\\ &\quad+\int_{\partial B_1}\{\bar{\mathbf{T}}^{(2)}\bar{\mathbf{u}}^{*(1)}-\bar{\mathbf{T}}^{(1)}\bar{\mathbf{u}}^{*(2)}\}\cdot\mathbf{n}\mathrm{d}A\\ &\quad+\int_{\partial B_1^c}\{\bar{\mathbf{t}}^{*(2)}\cdot\bar{\mathbf{u}}^{(1)}-\bar{\mathbf{t}}^{*(1)}\cdot\bar{\mathbf{u}}^{(2)}\}\mathrm{d}A.\end{aligned} \tag{48}$$

Next, from eq. (45), the divergence theorem and the boundary condition (47), we obtain

$$\begin{aligned}&\int_B [\xi^2\theta_0\gamma\{(\operatorname{div}\bar{\mathbf{u}}^{(1)})\bar{\theta}^{(2)}-(\operatorname{div}\bar{\mathbf{u}}^{(2)})\bar{\theta}^{(1)}\}+\rho\xi\{\bar{r}^{(2)}\bar{\theta}^{(1)}-\bar{r}^{(1)}\bar{\theta}^{(2)}\}]\mathrm{d}B\\ &=\int_{\partial B_2}\kappa^*\{\bar{\theta}^{*(2)}\nabla\bar{\theta}^{(1)}-\bar{\theta}^{*(1)}\nabla\bar{\theta}^{(2)}\}\cdot\mathbf{n}\mathrm{d}A+\int_{\partial B_2^c}\kappa^*\{\bar{\theta}^{(2)}\bar{q}^{*(1)}-\bar{\theta}^{(1)}\bar{q}^{*(2)}\}\mathrm{d}A.\end{aligned} \tag{49}$$

Using (49) in (48), we get

$$\int_B [\rho\theta_0\xi^2\{\bar{\mathbf{b}}^{(1)}\cdot\bar{\mathbf{u}}^{(2)} - \bar{\mathbf{b}}^{(2)}\cdot\bar{\mathbf{u}}^{(1)}\} + \rho\xi\{\bar{r}^{(2)}\bar{\theta}^{(1)} - \bar{r}^{(1)}\bar{\theta}^{(2)}\}]\mathrm{d}B$$

$$= \int_{\partial B_1} \theta_0\xi^2\{\bar{\mathbf{T}}^{(2)}\bar{\mathbf{u}}^{*(1)} - \bar{\mathbf{T}}^{(1)}\bar{\mathbf{u}}^{*(2)}\}\cdot\mathbf{n}\mathrm{d}A + \int_{\partial B_1^c} \theta_0\xi^2\{\bar{\mathbf{t}}^{*(2)}\cdot\bar{\mathbf{u}}^{(1)} - \bar{\mathbf{t}}^{*(1)}\cdot\bar{\mathbf{u}}^{(2)}\}\mathrm{d}A$$

$$+ \int_{\partial B_2} \kappa^*\{\bar{\theta}^{*(2)}\nabla\bar{\theta}^{(1)} - \bar{\theta}^{*(1)}\nabla\bar{\theta}^{(2)}\}\cdot\mathbf{n}\mathrm{d}A + \int_{\partial B_2^c} \kappa^*\{\bar{q}^{*(1)}\bar{\theta}^{(2)} - \bar{q}^{*(2)}\bar{\theta}^{(1)}\}\mathrm{d}A.$$

Inverting this equation by using the convolution theorem for Laplace transforms, we obtain the desired eq. (43). This completes the proof of the theorem.

References

[1] Biot M A, Thermoelasticity and irreversible thermodynamics, *J. Appl. Phys.* **27** (1956) 240–253

[2] Chandrasekharaiah D S, Thermoelasticity with Second Sound – A Review, *Appl. Mech. Rev.* **39** (1986) 355–376

[3] Chandrasekharaiah D S, A uniqueness theorem in the theory of thermoelasticity without energy dissipation, *J. Thermal Stresses* **19** (1996) 267–272

[4] Chandrasekharaiah D S, A Note on the uniqueness of solution in the linear theory of thermoelasticity without energy dissipation, *J. Elasticity,* **43** (1996) 279–283

[5] Dhaliwal R S and Singh A, *Dynamic Coupled Thermoelasticity* (Delhi: Hindustan Publ Corp.) (1980)

[6] Green A E and Naghdi P M, Thermoelasticity without energy dissipation, *J. Elasticity,* **31** (1993) 189–208

[7] Ionesco-Cazimir V, Problem of linear coupled thermoelasticity, *Bull. Poln. Sci. Techns.* **12** (1964) 473–480

[8] Parkus H, *Variational principles in thermo- and magnetoelasticity,* (Vienna: Springer-Verlag) (1972)

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 3, October 1998, pp. 217–226.

Rational curves on moduli spaces of vector bundles

SAMBAIAH KILARU

SPIC Mathematical Institute, 92, G N Chetty Road, Madras 600 017, India
e-mail: sambaiah@smi.ernet.in

MS received 17 March 1998; revised 12 May 1998

Abstract. We identify the spaces $\mathrm{Hom}_i(\mathbb{P}^1, M)$ for $i = 1, 2$, where M is the moduli space of vector bundles of rank 2 and determinant isomorphic to $\mathcal{O}(x_0), x_0 \in X$, on a compact Riemann surface of genus $g \geq 2$.

Keywords. Riemann surfaces; determinant bundles; Hilbert scheme; jumping divisor.

1. Introduction

Let X be a compact Riemann surface of genus $g \geq 2$ and M be the moduli space of stable vector bundles of rank 2 and fixed determinant on X. We study the structure of the space $\mathrm{Hom}_i(\mathbb{P}^1, M)$, $i = 1, 2$ of maps of degree one and two from $\mathbb{P}^1$ into M. This study draws its origin from attempts to compute the quantum cohomology of the moduli space M which has recently become an important topic for research (see for example [VM] and [D]).

We now outline the main results of the paper. Consider the natural vector bundle of extensions on the Picard variety $\mathrm{Pic}^0(X)$. The content of Theorem 8 and the discussion following it is that the space $\mathrm{Hom}_1(\mathbb{P}^1, M)$ is the total space of a principal $PGL(2)$-bundle on the Grassmannization of this bundle. In the case of degree two maps, one of the components of the scheme $\mathrm{Hom}_2(\mathbb{P}^1, M)$ is shown to be naturally related to the moduli space of stable bundles of rank two and determinant isomorphic to $\mathcal{O}(x - x_0), x \in X$ (Theorem 9). It is only this component which will figure in the definition of the quantum cohomology of M.

The paper is laid out as follows § 2 lists notation and terminology for subsequent use. In § 3 we collect together various lemmas of technical nature on the determinant of cohomology of a family of vector bundles. Section 4 consists of constructions related to elementary transformations of vector bundles along divisors. These form the main technical framework of the paper. The next two sections deal with the study of maps of degree one and two respectively.

2. Notation

- If E is a vector bundle on $X \times Y$, we denote by $E_x (x \in X)$ the bundle over Y gotten by restricting E to $\{x\} \times Y$.
- If E is a rank n bundle on $X, \lambda(F) := \Lambda^n F$.
- If X and Y are varieties then p and q denote the projections of $X \times Y$ on X and Y respectively.

- $\mathcal{SU}(n, d)$ denotes the moduli space of stable bundles of rank n and degree d.
- For Y smooth, T_Y denotes the tangent bundle of Y and K_Y denotes λT_Y^*.
- $\mathrm{Pic}(X)$ denotes the Picard group of X.
- If $x \in X$, $k(x)$ stands for the torsion sheaf of height 1 supported at x.

3. Preliminaries

Let X be a compact Riemann surface of genus $g, g \geq 2$. Fix a point $x_0 \in X$. Let $M := \mathcal{SU}(2, \mathcal{O}(x_0))$ denote the moduli space of stable bundles on X of rank 2 with determinant isomorphic to $\mathcal{O}(x_0)$. It is well-known that there are Poincaré families on $X \times M$. It is also known that $\mathrm{Pic}(M)$ is $\mathbb{Z}$ and let u denote the ample generator of $\mathrm{Pic}(M)$.

By [R], there is a unique Poincaré bundle E on $X \times M$ with the property that $\det E_{x_0}$ is isomorphic to the line bundle u.

We call such an E *the rigidified Poincaré bundle* and note that $\det E_x$ is independent of $x \in X$.

In the following we recall the definition of the determinant line bundle on M associated to E and a couple of lemmas of technical nature.

Let $\mathcal{F} = \{\mathcal{F}_t\}_{t \in T}$ be a family of vector bundles on X parametrised by a complex manifold T.

If $q : X \times T \to T$ is the natural projection, $R^i q * \mathcal{F}$ (f or $i = 0, 1$) is given globally on T by the cohomology of a complex $j : K^0 \to K^1$ where K^i are locally free and coherent.

DEFINITION 1.

The line bundle $D(T, \mathcal{F})$ on T, called the determinant of cohomology of $\mathcal{F}$ with respect to T, is defined as $(\lambda K^0)^{-1} \otimes (\lambda K^1)$ where λ is the highest exterior power.

Since $\mathrm{Pic}(M) \simeq Z$, $D(M, E) \simeq u^l$ for some $l \in \mathbb{Z}$. It can be shown that ([R] proof of theorem 1) $D(M, E \otimes p^* V) \simeq u$ where V is any rank 2 degree $2g - 3$ bundle on X.

Remark. The Poincaré bundle E has the property that $D(M, E \otimes q^* V) \simeq \det E_{x_0}$ where V is as above.

Suppose $\mathcal{F}$ is a bundle on $X \times T$, where T is a complex manifold, such that $\mathcal{F} \simeq p^* V \otimes q^* H$ where V is a vector bundle on X, H is a vector bundle on T. Then $D(T, \mathcal{F})$ can be expressed in terms of V and H.

Lemma 2. $D(T, \mathcal{F}) \simeq (\det H)^{-\chi(V)}$.

Proof. By the projection formula, for $i = 0, 1$

$$R^i q_* (p^* V \otimes q^* H) = R^i q_* (p^* V) \otimes H,$$

hence, $D(T, \mathcal{F}) \simeq \det H)^{-\chi(V)}$.

DEFINITION 3.

A holomorphic map $\psi : \mathbb{P}^1 \to M$ is said to be of degree d if $D(\mathbb{P}^1, (id \times \psi)^* \ (E \otimes p^* V))$ is isomorphic to $\mathcal{O}(d)$, where V is a rank 2 degree $2g - 3$ bundle on X.

Remark. Note that E on $X \times M$ is rigidified implies that $D(M, E \otimes p^* V) \simeq \lambda(E_x)$. Now the fact that the determinant of cohomology commutes with base change implies that we

have an isomorphism:

$$D(\mathbb{P}^1, (id_X \times \psi)^*(E \otimes p^*V)) \simeq \lambda(\psi^* E_x).$$

Lemma 4. *$\psi : \mathbb{P}^1 \to M$ is a holomorphic map of degree d if and only if* deg $D(\mathbb{P}^1,$ *$(id \times \psi)^* E) = (g-1)d$.*

Proof. Let $q : X \times \mathbb{P}^1 \to \mathbb{P}^1$ be the natural projection and let $\mathrm{d}(E)$ denote $D(\mathbb{P}^1, E)$. Let F and G be holomorphic vector bundles on $X \times \mathbb{P}^1$ of ranks r and n respectively. Then

$$(*) \cdot \mathrm{d}(F \otimes G) \cdot \mathrm{d}(F)^{-n} \cdot \mathrm{d}(G)^{-r} = \cdot \mathrm{d}(\lambda(F)\lambda(G)) \cdot \mathrm{d}(\lambda(F))^{-1} \cdot \mathrm{d}(\lambda(G))^{-1}$$

in $\mathrm{Pic}(\mathbb{P}^1)$. The proof of this can be deduced from the Grothendieck–Riemann–Roch theorem (see [B-R] Lemma 4.12). In the above formula, set $F = (id \times \psi)^* E$ and $G = p^* V$, where V is a rank 2 degree $2g-3$ bundle on X.

$\Rightarrow$: We first note that $\lambda(F) \simeq p^*\mathcal{O}(x_0) \otimes q^*\mathcal{O}(d)$ and that $\lambda(G) \simeq p^* L$, where L is a line bundle of degree $2g-3$. Substituting these in $(*)$ and using lemma 2, we easily see that $\deg{\cdot}\mathrm{d}(F) = \mathrm{d}(g-1)$.

$\Leftarrow$: Let $\mathrm{d}(F \otimes G)$ be $\mathcal{O}(r)$, for some $r \in \mathbb{Z}$. From the remark above, we have $\lambda(F) \simeq p^*(x_0) \otimes q^*\mathcal{O}(r)$. We can now use $(*)$ to evaluate r to be d.

We next need a result concerning the H–N filtration of vector bundles. We need to know that the H–N filtration 'globalizes' correctly in a family of vector bundles. For a proof of this lemma, see [L] proposition 11.1.2. This, in our case, can also be proved in an elementary fashion by slightly modifying the proof of Prop. 15.4 of [A–B].

Lemma 5. *Let $\mathcal{E}$ be a holomorphic bundle on $X \times C$ where C is a compact Riemann surface. Suppose $\mathcal{E}_x$ has the same H–N type for every $x \in X$. Then there is a holomorphic filtration of the bundle $\mathcal{E}$ by subbundles over $X \times C$ such that it induces the H–N filtration on each $\mathcal{E}_x$, $x \in X$.*

4. The jumping divisor

Let X be a compact Riemann surface (in this section alone we will assume it to be of arbitrary genus) and F be a rank 2 holomorphic bundle on $X \times \mathbb{P}^1$ such that $\det F_x$ is isomorphic to $\mathcal{O}(d)$ for all $x \in X$ and for some fixed $d > 0$.

A theorem of Grothendieck implies that $F_x \simeq \mathcal{O}(l_x) \oplus \mathcal{O}(-l_x + d)$ where we assume $l_x \geq -l_x + d$. Since $d > 0$, $l_x > 0$. Let $l = \min\{l_x / x \in X\}$. Then $h^0(\mathbb{P}^1, F_x \otimes \mathcal{O}(-l-1)) = 0$ at some $x \in X$. By the theorem of semicontinuity, the set U of all points $x' \in X$ such that $h^0(\mathbb{P}^1, F_{x'} \otimes \mathcal{O}(-l-1)) = 0$, is open. This in turn implies that for all $y \in U$, $F_y \otimes \mathcal{O}(-l-1) \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-2l+d-1)$. Hence U is the largest open set such that for $x \in U$, $F_x \simeq \mathcal{O}(l) \oplus \mathcal{O}(-l+d)$.

Let the points in $X - U$ be $x_1, x_2, \ldots, x_n$. Clearly $F_{x_i} \simeq \mathcal{O}(l_{x_i}) \oplus \mathcal{O}(-l_{x_i} + d)$ where $l_{x_i} > l$.

The local moduli of the bundle F_{x_i} is the germ of the vector space $H^1(\mathbb{P}^1 \text{ End } F_{x_i})$ at the origin, and the differential of the classifying map from analytic open sets V_{x_i} of x_i, to the local moduli of F_{x_i} is the Kodaira–Spencer map at x_i for the family F, when thought of as a family of vector bundles on $\mathbb{P}^1$ parametrised by X. Fixing a basis for the vector space $H^1(\mathbb{P}^1, \text{End } F_{x_i})$, the classifying map is given by a coordinate-wise power series. Let $m_{i,1} \cdots m_{i,r}$ be non-negative integers such that $m_{i,j}$ is the first non-zero coefficient in the jth coordinate of this power series. Define $m_i = \min\{m_{i,1}, m_{i,2} \cdots m_{i,r}\}$.

DEFINITION 6.

The jumping divisor of F on X is defined as the divisor $D = \sum_{i=1}^{n} m_i x_i$.

Lemma 7. There exists a bundle F' on $X \times \mathbb{P}^1$ along with a map $F' \to F$ which is an isomorphism outside the divisor D, and such that jumping divisor of F' on X is the zero divisor.

Proof. Let $D = \sum_{i=1}^{r} m_i x_i$ be the jumping divisor of F. That is, $F_{x_i} \simeq \mathcal{O}(l_i^0) \oplus \mathcal{O}(-l_i^0 + d)$ and $l_i^0 > 1$.

Consider the vector space $\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(l_i^0), \mathcal{O}(-l_i^0 + d))$ and the universal extension

$$0 \to p^*\mathcal{O}(-l_i^0 + d) \to \mathcal{V}_i \to p^*\mathcal{O}(l_i^0) \to 0 \tag{1}$$

on $\mathrm{Ext}^1 \times \mathbb{P}^1$. It is not hard to see that $\mathcal{V}_i$ is a locally complete family of deformations of F_{x_i}. (In fact, the Kodaira Spencer map for $\mathcal{V}_i$ at the origin of Ext^1 is the natural identification of Ext^1 with $H^1(\mathbb{P}^1, \mathrm{End}\, F_{x_i})$). Hence there are analytic open sets U_i of x_i and local classifying maps ψ_i from U_i to Ext^1, such that $(id \times \psi_i)^* \mathcal{V}_i \simeq F|_{U_i \times \mathbb{P}^1}$. Denote by D_i the scheme Spec $\mathcal{O}_{X,x_i}/\mathcal{M}_{x_i}^{m_i}$ and by τ_i the torsion sheaf $\mathcal{O}_{x_i}/\mathcal{M}_{x_i}^{m_i}$ on X. Since V_i is split over $\{0\} \times \mathbb{P}^1, \psi_i^*(1)$ is split on $D_i \times \mathbb{P}^1$. Let $\varrho_i : F|_{D_i \times \mathbb{P}^1} \to q^*\mathcal{O}(-l_i^0 + d) \to 0$ be one such splitting. Let τ denote the torsion sheaf

$$p^*\tau_1 \otimes q^*\mathcal{O}(-l_{x_1}^0 + d) \oplus p^*\tau_2 \otimes q^*\mathcal{O}(-l_{x_2}^0 + d) \oplus \cdots \oplus p^*\tau_r \otimes q^*\mathcal{O}(-l_{x_r}^0 + d).$$

Let ϕ be the surjection

$$F \to \tau \to 0$$

of $\mathcal{O}_{X \times \mathbb{P}^1}$ modules obtained by extending $(\varrho_1, \varrho_2, \ldots, \varrho_r)$ by zero outside the divisor $D \times \mathbb{P}^1$. Set $F_1 := \ker \phi$. Then it can be checked that F_1 is a locally free sheaf on $X \times \mathbb{P}^1$. Restricting the short exact sequence

$$0 \to F_1 \to F \to \tau \to 0$$

to $\{x_i\} \times \mathbb{P}^1$, we have following exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-l_i^0 + d) \arrow[r] & F_{1_{x_i}} \arrow[rr] \arrow[dr] & & \mathcal{O}(l_i^0) \oplus \mathcal{O}(-l_i^0 + d) \arrow[r] & \mathcal{O}(-l_i^0 + d) \arrow[r] & 0 \\
& & & \mathcal{O}(l_i^0) \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Claim: $F_{1_{x_i}}$ is *non-split* extension of $\mathcal{O}(l_i^0)$ by $\mathcal{O}(-l_i^0 + d)$. (See [RF] pages 132–134).

Write $F_{1_{x_i}} \simeq \mathcal{O}(l_i^1) \oplus \mathcal{O}(-l_i^1 + d)$

We have non-split exact sequence of bundles

$$0 \to \mathcal{O}(-l_i^0 + d) \to \mathcal{O}(l_i^1) \oplus \mathcal{O}(-l_i^1 + d) \to \mathcal{O}(-l_i^0) \to 0$$

This forces $l_i^1 < l_i^0$.

The jumping divisor for F_1 be $D_1 = \sum_{i=1}^{r} m_i^1 x_i$. Then applying the above process successively, after a finite number of steps we get a bundle $F' := F_n$ such that the jumping divisor of F' is the zero divisor.

5. Degree one maps[1]

In this section we study degree 1 maps from $\mathbb{P}^1$ to M and identify the space of degree 1 maps as the $PGL(2)$ bundle over G, where G is a Grassmann bundle over $\mathrm{Pic}^0(X)$.

Let $L \in \mathrm{Pic}^0(X)$, we know that a non-split extension of $L^{-1}(x_0)$ by L is a stable bundle and that any two extensions which differ by a non-zero scalar are isomorphic. The family of non-split extensions of $L^{-1}(x_0)$ by L modulo non-zero scalars is parametrized by $\mathbb{P}(H^1(X, \mathrm{Hom}(L^{-1}(x_0), L)))$. For $L \in \mathrm{Pic}^0(X)$, we denote this projective space by $\mathbb{P}_{\mathcal{L}}$. By [R], the induced map from $\mathbb{P}_L$ to M induces an isomorphism from $\mathrm{Pic}(M)$ to $\mathrm{Pic}(\mathbb{P}_L)$. There is a universal family of extensions on $X \times \mathbb{P}_L$:

$$0 \to p^*L \otimes q^*\mathcal{O}(1) \to V \to p^*L^{-1}(x_0) \to 0. \tag{$\dagger$}$$

It is also clear that V is the pull back of the rigidified Poincaré bundle.

Let $\mathbb{P}^1$ to $\mathbb{P}_L$ be a linear embedding. Restricting the exact sequence ($\dagger$) to $\mathbb{P}^1$, we have a rank 2 family on $X \times \mathbb{P}^1$. Therefore it induces a map from $\mathbb{P}^1$ to M. By a simple calculation of the determinant of cohomology of V, we can conclude that this map from $\mathbb{P}^1$ to M induced by the family V is of degree 1. We shall prove that every degree 1 map is of this form:

Theorem 8. *Let $\psi : \mathbb{P}^1 \to M$ be a degree one map. Then it factors as $\mathbb{P}^1 \to \mathbb{P}_L \to M$ for some $L \in \mathrm{Pic}^0(X)$ where $\mathbb{P}^1$ to $\mathbb{P}_L$ is a linear embedding.*

Proof. On $X \times \mathbb{P}^1$ set $F = (id \times \psi)^*E$.

Claim: The jumping divisor of F is the zero divisor.

Suppose that the claim is true. Then $F_p \simeq \mathcal{O}(l) \oplus \mathcal{O}(-l+1)$ and $l > 0$.

By lemma 5, there is a line sub-bundle $\mathcal{L}$ of F. Since $\mathcal{L}_x \simeq \mathcal{O}(l)$ for all $x \in X$, $\mathcal{L} \simeq p^*L \otimes q^*\mathcal{O}(l)$ where L is a line bundle on X of degree, say d. Hence there is an exact sequence on $X \times \mathbb{P}^1$:

$$0 \to p^*L \otimes q^*\mathcal{O}(l) \to F \to p^*L^{-1}(x_0) \otimes q^*\mathcal{O}(-l+1) \to 0.$$

If we take the determinant of cohomology of F, then

$$D(\mathbb{P}^1, F) \simeq D(\mathbb{P}^1, p^*L \otimes q^*\mathcal{O}(l)) \otimes D(\mathbb{P}^1, p^*L^{-1}(x_0) \otimes q^*\mathcal{O}(-l+1))$$

Equating the degrees of the line bundles in the above isomorphism, we have

$$g - 1 = l(g-d-1) + (l-1)(2-g-d).$$

This implies $-2dl + d + l = 1$. Since l is a positive integer and d is an integer, we have $d = 0$ and $l = 1$.

It only remains to prove the claim. Let the jumping divisor of F be $\sum_{i=1}^n m_i x_i$ such that

$$F_{x_i} \simeq \mathcal{O}(l_i) \oplus (-l_i + 1).$$

Applying lemma 7, we have to locally free sheaf F' on $X \times \mathbb{P}^1$ with the property that $F'_x \simeq \mathcal{O}(l) \oplus \mathcal{O}(-l+1)$ for all $x \in X$. Then, from lemma 5 we have:

$$0 \to p^*L_1 \otimes q^*\mathcal{O}(l) \to F' \to p^*L_2 \otimes q^*\mathcal{O}(-l+1) \to 0$$

[1] We came across the preprint [VM] after this work was done. Although the results of this section are also found in § 3 of [VM], proof techniques are quite different.

If the degree of L_1 is d, then the degree of L_2 is of degree $1-d-\sum_j\sum_i m_i^j$ since the degree of F'_p is $1-\sum_j\sum_i m_i^j$.

Therefore $D(\mathbb{P}^1, F') \simeq \mathcal{O}(l)^{-\chi(L_1)} \otimes \mathcal{O}(-l+1)^{-\chi(L_2)}$.

Hence the degree of the bundle $D(\mathbb{P}^1, F')$ is equal to

$$l+d+g-2-2dl-\sum_j\sum_i m_i^j l+\sum_j\sum_i m_i^j. \tag{1}$$

On the other hand, we can compute the degree of the determinantal bundle of F' by going back to the definition of F' (cf. lemma 7). It is

$$g-1-\sum_i\sum_i m_i^j(l_i^j-1). \tag{2}$$

Equating (1) and (2) we have:

$$l+d-2dl+\sum_j\sum_i m_i^j(l_i^j-l)-1=0.$$

But since $l_i^j > l$, the above equation can be written as

$$l-2dl+d-1<0.$$

Hence $d > 1-l/(1-2l)$, as $l>0$.

Since F'_p is semi-stable of degree $1-\sum_j\sum_i m_i^j$ and L_1 is a sub-bundle of degree $d\geq 1$, this is a contradiction to the semi-stability of F'_p. This proves the claim.

Let us summarise what we have done in terms of the moduli of degree one maps from $\mathbb{P}^1$ to M. First note that if $f:\mathbb{P}^1 \to M$ is a degree 1 map and $\phi \in \mathrm{Aut}(\mathbb{P}^1)$, then $fo\phi$ is again a degree 1 map. Let $\mathcal{L}$ be the rigidified Poincaré bundle on $X\times \mathrm{Pic}^0(X)$ rigidified at x_0. Then $R^1q*(\mathcal{L}^2(-x_0))$ is a locally free sheaf on $\mathrm{Pic}^0(X)$ of rank g. Take the associated projective bundle and call it P. Let G be the Grassmann bundle of lines associated to the projective bundle P over $\mathrm{Pic}^0(X)$. Then G comes with a natural $\mathbb{P}^1$ bundle over it. Denote by S the total space of the natural principal $PGL(2)$ bundle associated to this. We have shown *set-theoretically* that this is the space of degree 1 maps. A simple calculation shows that the dimension of S is $3g-1$.

Let $\mathrm{Hom}_d(\mathbb{P}^1, M)$ denote the set of degree d morphisms from $\mathbb{P}^1$ to M. It can be seen to be an open subscheme of $\mathrm{Hil}\,b_P(\mathbb{P}^1\times M)$, where the polynomial P is the Hilbert polynomial of the graph of a degree d map with respect to the polarisation $\mathcal{O}(1)\otimes u$ on $\mathbb{P}^1\times M$ (see [K]).

The expected dimension of $\mathrm{Hom}_d(\mathbb{P}^1, M)$ at f equals $\deg(f^*T_M)+\dim M$ and it is attained if $H^1(\mathbb{P}^1, f^*T_M)=0$. See [K] for details. We shall now show that $\mathrm{Hom}_1(\mathbb{P}^1, M)$ is smooth of dimension $=-\deg(f^*K_M)+3g-3=3g-1$ (since one knows that $-K_M\simeq u^2$ on M (see [R]) and $f^*u\simeq \mathcal{O}_{\mathbb{P}^1}(1)$).

To compute $h^1(\mathbb{P}^1, f^*T_M)$, first of all note that

$$T_M \simeq R^1q*(adE)$$

where adE is the bundle of traceless endomorphisms of the rigidified Poincaré bundle on $X\times M$. So if $F=(1\times f)^*E$, $f^*T_M\simeq R^1q*(adF)$. So we want to compute $h^1(\mathbb{P}^1, R^1q_*adF)$. But since $R^iq_*adF=0\,\forall i\neq 1$, the Leray spectral sequence associated to the projection p degenerates at E_2 and we have

$$H^1(\mathbb{P}^1, R^1p_*adF)\simeq H^2(X\times\mathbb{P}^1, adF).$$

Consider now $R^i p_* adF$. Since $F_x \simeq \mathcal{O} \oplus \mathcal{O}(1) \forall x \in X$, we see easily that $R^i p_* adF = 0 \forall i \geq 1$. Hence by the Leray spectral sequence for the projection p, we have:

$$H^2(X \times \mathbb{P}^1,\ adF) \simeq H^2(X, q_*(adF)) = 0.$$

Since the family of degree 1 maps parametrised by the space S is injective, the classifying map from S to $\mathrm{Hom}_1(\mathbb{P}^1, M)$ is bijective. But as the dimension of S equals the dimension of $\mathrm{Hom}_1(\mathbb{P}^1, M)$, it is an isomorphism.

When the genus of X is 2, the work [N-R 1] shows that the space of lines in the moduli space can be identified with $\mathrm{Pic}^0(X)$ which is the $PGL(2)-$quotient of $\mathrm{Hom}_2(\mathbb{P}^1, M)$.

6. Degree two maps

In this section we shall study degree 2 maps from $\mathbb{P}^1$ to M and identify a component of the $\mathrm{Hom}_2(\mathbb{P}^1, M)$ as the variety $\mathcal{SU}(2, \mathcal{O}(x - x_0))x \in X$.

If $F = (1 \times f)^* E$ and $f \in \mathrm{Hom}_2(\mathbb{P}^1, M)$, then Case (1): $F_x \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ at all but finitely many $x \in X$, Case (2): $F_x \simeq \mathcal{O}(l) \oplus \mathcal{O}(-l+2)$ (where $l > 1$) at all but finitely many $x \in X$.

Case 1.

Let V be a rank 2 stable bundle on X with determinant isomorphic to $\mathcal{O}(x - x_0)$. Consider the following tautological surjection of sheaves of modules on $X \times \mathbb{P}(V_x^*)$ for some $x \in X$:

$$p^* V^* \otimes q^* \mathcal{O}(-1) \to p^* k(x) \to 0$$

Let V_1^* denote the kernel of the above map. We write V_1 for the dual of V_1^*. Then V_1 is a locally free sheaf on $X \times \mathbb{P}^1$. It is easy to see that V_{1_t} $(\forall t \in \mathbb{P}^1)$ is stable of degree 1 and that $D(\mathbb{P}^1, V_1 \otimes p^* F) \simeq \mathcal{O}(2)$, where F' is any rank 2 degree $2g - 3$ bundle on X.

Therefore, we have a degree 2 map $\psi : \mathbb{P}^1 \to M$ given by the family V. We shall prove in the following theorem that every degree 2 map which falls under case (1) is of this form.

Theorem 9. *Let $\psi : \mathbb{P}^1 \to M$ be a degree 2 map such that $(id \times \psi)^* E_{x \times \mathbb{P}^1}$ is generically isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Then there is a stable bundle V on X of rank 2 and determinant $\simeq \mathcal{O}(x - x_0)$ for some $x \in X$ such that*

$$0 \to p^* V \otimes q^* \mathcal{O}(1) \to (id \times \psi)^* E \to p^* k(x) \to 0$$

Proof.

Write F for $(id \times \psi)^* E$ and let the jumping divisor of F be D.

Claim: $D = (x)$ for some $x \in X$ and $F_x \simeq \mathcal{O}(2) \oplus \mathcal{O}$.

Suppose that the claim is true. Consider the following map of sheaves of $\mathcal{O}_{X \times \mathbb{P}^1}$ modules

$$F \to p^* k(x) \to 0.$$

Let F' denote the kernel. Then F' is locally free and $F'_x \simeq \mathcal{O}(1) \oplus \mathcal{O}(1) \forall x \in X$. So $F' \simeq p^* V \otimes q^* \mathcal{O}(1)$.

Clearly $\det(V) \simeq \mathcal{O}(x - x_0)$. It can be seen easily that V is semi-stable. We prove that V is in fact stable.

Suppose that V is semi-stable but not stable. There exists a line sub-bundle L of V of degree 0. Then $p^*L \otimes q^*\mathcal{O}(1)$ is a sub-bundle of F' and if the image sheaf of $p^*L \otimes q^*\mathcal{O}(1)$ in F is a sub-bundle then, we have an exact sequence of vector bundles on $X \times \mathbb{P}^1$:

$$0 \to p^*L \otimes q^*\mathcal{O}(1) \to F \to p^*L^{-1}(x_0) \otimes q^*\mathcal{O}(1) \to 0 \tag{1}$$

Restricting the exact sequence (1) to $\{x\} \times \mathbb{P}^1$, we have

$$0 \to \mathcal{O}(1) \to \mathcal{O}(2) \oplus \mathcal{O} \to \mathcal{O}(1) \to 0$$

which is impossible. That is, the sheaf inclusion of $p^*L \otimes \mathcal{O}(1)$ in F goes down in rank at some point $(x,p) \in X \times \mathbb{P}^1$. If L' denotes the sub-bundle of F_p generated by the image of L in F_p, then the degree of L' is strictly bigger than zero. This contradicts the stability of F_p. Hence V is *stable* of degree 0.

To prove the claim, let the jumping divisor of F be $D = \sum_{i=1}^n m_i x_i$ with $F_{x_i} \simeq \mathcal{O}(l_i) \oplus \mathcal{O}(-l_i+2)$. Notice that D is not the zero divisor. Applying lemma 7, we have as in the lemma, a bundle F' with $F'_x \simeq \mathcal{O}(1) \oplus \mathcal{O}(1) \forall x \in X$.

Degree of F'_p is $1 - \sum_j \sum_i m_i^j$. Hence it can be seen that $F' \simeq p^*V \otimes q^*\mathcal{O}(1)$ for some vector bundle V on X of degree $1 - \sum_j \sum_i m_i^j$. From this we can compute the degree of $D(\mathbb{P}^1, F')$. It is equal to

$$2g - 3 + \sum_j \sum_i m_i^j \tag{2}$$

On the other hand, we can compute the degree of $D(\mathbb{P}^1, F')$ by going back to the definition of F' (cf. proof of lemma 7). It comes out to be

$$2g - 2 + \sum_j \sum_i m_i^j (2 - l_i^j) \tag{3}$$

Equating (2) and (3), we have

$$\sum_j \sum_i m_i^j (1 - l_i^j) + 1 = 0.$$

Since we know that $l_i^j \geq 2$ and $m_i^j \geq 1$, the above equality is true only when $m_i = 0$ for $i \geq 2$ and $l_1^0 = 2$.

Hence the theorem.

Let M' denote the inverse image of the copy of X in $\mathrm{Pic}^0(X)$ (defined by sending x to $\mathcal{O}(x - x_0)$ under the determinant map

$$\Lambda : \mathcal{SU}(2,0) \to \mathrm{Pic}^0(X)$$

where $\mathcal{SU}(2,0)$ stands for the moduli space of stable bundles on X of rank 2 and degree 0. Note that M' is smooth of dimension $3g - 2$. Let Q denote the natural $PGL(2)$-principal bundle on $X \times M'$ associated to the projective Poincaré bundle on $X \times M'$. Consider the map $(id \times \Lambda) : X \times M' \to X \times X$. Note that the natural projection of $(id \times \Lambda)^{-1}(\Delta)$ on M' is an isomorphism. Let S_1 stand for the pull-back of $Q|(id \times \Lambda^{-1}(\Delta))$ to M' under the inverse of this isomorphism.

On the other hand, if $f \in \mathrm{Hom}_2(\mathbb{P}^1, M)$ is as in case (1), then a spectral sequence argument (along the lines of discussion which follows the proof of theorem 8) shows that the scheme $\mathrm{Hom}_2(\mathbb{P}^1, M)$ is smooth of dimension $3g + 1$ at f. It we now note that S_1 is

injectively parameterized, we get an injective map from S_1 to $\mathrm{Hom}_2(\mathbb{P}^1, M)$ whose image consists of points which come under case (1). This implies that such points form an irreducible component of the scheme $\mathrm{Hom}_2(\mathbb{P}^1, M)$ isomorphic to S_1. This is closely related to the work [N-R 2]. We shall consider this connection in a later work.

Case 2.

Theorem 10. *Let $\psi : \mathbb{P}^1 \to M$ be a degree 2 map such that $(Id \times \psi)^* E$ is generically $\mathcal{O}(l) \oplus \mathcal{O}(-l+2)$ for some $l > 1$. Then the bundle $(1 \times \psi)^* E$ can be expressed as*

$$0 \to p^* L \otimes q^* \mathcal{O}(2) \to (1 \times \psi)^* E \to p^* L^{-1}(x_0) \to 0$$

for some line bundle L on X of degree 0.

Proof. We denote by F the bundle $(id \times \psi)^* E$.

Arguing along the lines of the proofs of lemmas 8 and 9, we can conclude that the jumping divisor of F is the zero divisor.

Hence $F_x \simeq \mathcal{O}(l) \oplus \mathcal{O}(-l+2)$ for every $x \in X$. Applying the lemma 7 we have an exact sequence of vector bundles

$$0 \to L' \to F \to L'' \to 0.$$

We know that $L'_x \simeq \mathcal{O}(2) \forall x \in X$. Hence we can conclude that $L' \simeq p^* L \otimes q^* \mathcal{O}(2)$ for some line bundle L on X.

Similarly we can conclude that $L'' \simeq p^* L^{-1}(x_0)$.

Hence, rewriting the above exact sequence, we have:

$$0 \to p^* L \otimes q^* \mathcal{O}(2) \to F \to p^* L^{-1}(x_0) \to 0.$$

Now arguing again as in the proof of theorem 8, it can be shown that degree of L is 0. Hence the theorem.

Acknowledgements

The author thanks P A Vishwanath and V Balaji for many discussions and constant encouragement. He also thanks Prof. M S Narasimhan for suggesting the problem which led to this work.

References

[A-B] M F Atiyah and R Bott, The Yang–Mills equations over Riemann surfaces, *Phil. Trans. R. Soc. Lond.* **A308** (1982), 523–615

[B-R] I Biswas and N Raghavendra, Determinants of parabolic bundles on Riemann surfaces, *Proc. Indian Acad. Sci.* **103**, No. 1, (1993), 43–71

[D] S K Donaldson, Floer homology and algebraic geometry, Vector bundles in algebraic geometry, *London Math. Soc. Lecture Notes Series*, **208** (Cambridge University Press, Cambridge) (1995), 119–138

[K] Janos Kollar, *Rational Curves on Algebraic Varieties*, (Springer-Verlag) (1996) pp. 92–93

[L] J Le Potier, Lectures on Vector Bundles, (Cambridge University Press) (1997), 165–167

[N-R 1] M S Narasimhan and S Ramanan, Moduli of vector bundles on a compact riemann surface, *Ann. of Math.* **89** (1969), 19–51

[N-R 2] M S Narasimhan and S Ramanan, Geometry of Hecke Cycles-1, C *P Ramanujam – A Tribute.* (1978), 291–345

[R] S Ramanan, The moduli spaces of vector bundles over an algebraic curve, *Math. Ann.* **200** (1973), 69–84

[RF] Robert Friedman, Vector bundles and $SO(3)$-invariants for elliptic surfaces, *J. Amer. Math. Soc.* **8**, No. 1, Jan. (1995), 29–139

[VM] Vicente Munoz, Quantum cohomology of the moduli space of stable vector bundles over a Riemann surface, alg-geom/9711030

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 3, October 1998, pp. 227–249.

The Chow ring of a singular surface

J G BISWAS and V SRINIVAS

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India
e-mail: jishnu@math.tifr.res.in; srinivas@math.tifr.res.in

MS received 5 January 1998; revised 8 May 1998

Abstract. We construct the Chow ring $CH^*(X) = CH^0(X) \oplus CH^1(X) \oplus CH^2(X)$ of a reduced, quasi-projective surface X, together with Chern class maps $c_i : K_0(X) \to CH^i(X)$, with the usual properties. As a consequence, we show that the cycle map $CH^2(X) \to F_0K_0(X)$ is an isomorphism. Our treatment is greatly influenced by an unpublished 1983 preprint of Levine's, but is much simpler, since we deal only with surfaces.

Keywords. Chow ring; singular surface; Chern class maps.

1. Introduction

Let X be a reduced, quasi-projective variety over an algebraically closed field k, with $\dim X = 2$. Our aim here is to construct the Chow ring $CH^*(X) = CH^0(X) \oplus CH^1(X) \oplus CH^2(X)$, and Chern class maps $c_i : K_0(X) \to CH^i(X)$, with the usual properties, and finally to prove that if $F_0K_0(X)$ is the subgroup of $K_0(X)$ generated by the classes of smooth points (of codimension 2), then $c_2 : F_0K_0(X) \to CH^2(X)$ is an isomorphism, inverse (up to sign) to the natural surjection (the *cycle map*) $CH^2(X) \to F_0K_0(X)$. In particular, the cycle map $CH^2(X) \to F_0K_0(X)$ is an isomorphism.

Our discussion is greatly influenced by an old unpublished manuscript [L1] of Levine's, except that simplifications are possible in our situation – the only subtle issues involved relate to cycles of codimension 2, and the only intersection products we need to consider are those of Cartier divisors. Inspite of this, the reader will observe that the discussion is fairly technical; that in [L1] is even more so. However, a point to note is that our treatment (as also [L1]) uses only techniques from algebraic geometry, and does not need results and techniques of algebraic K-theory.

This paper is meant as a complement to another paper [BiS] of ours on Roitman's Theorem for singular projective varieties over $\mathbb{C}$. For the purposes of [BiS], what is needed is the assertion that the cycle map $CH^2(X) \to F_0K_0(X)$ is an isomorphism, but we need to develop the remaining machinery in order to prove this. We do this to fill a gap in the literature, since [L1] is still unpublished, and (as far as we know) is not yet in final form. We note also that the result that $CH^2(X) \to F_0K_0(X)$ is an isomorphism has been used in the proof of the main result of [L].

We remark here that some of this ground is also covered in [LW]; for example, apart from introducing the notion of a Cartier curve, [LW] contains a direct proof that there is a well-defined cycle map $CH^2(X) \to K_0(X)$; also, there is a proof of the desired injectivity of the cycle map when X is *affine* (these ideas go back to the paper [MS] of Murthy and Swan). Other cases treated earlier include work of Collino [Co] (isolated singularities)

and Pedrini–Weibel [PW] (singular locus contained in an affine). However, as far as we know, the general case, and the case of projective surfaces (which we need), is treated only in [L1].

The ground field k will be fixed throughout our discussion; it is understood that schemes and morphisms considered will be k-schemes and k-morphisms respectively, and all products will be relative to k; projective spaces, Grassmannians etc. will be over k, and the terms 'projective variety', 'projective scheme' etc. will be understood to mean 'projective k-variety', 'projective k-scheme' etc.

We make use of the following notation: if Y is a closed subscheme of codimension i of a variety Z, let $[Y]$ denote the cycle of codimension i associated to Y. We recall that this is defined as follows: if $Y_1, \ldots, Y_r$ are the irreducible components of Y_{red} which have codimension equal to i, and if $y_j \in Y_j$ is the generic point, then $[Y] = \sum_{j=1}^{r} n_j[Y_j]$, where n_j is the length of the Artinian local ring $\mathcal{O}_{y_j,Y}$. In particular, if Y has 0-dimensional support, then $[Y]$ is a 0-cycle. All facts cited about cycle theory and Chern classes on non-singular varieties, as well as the definitions of operations on cycles (like proper push-forward), are in Fulton's book [Ful].

Finally, X will always denote a reduced quasi-projective surface.

2. Definition of the Chow ring

As in [BiS], $X_{\rm sing}$ denotes the reduced subscheme consisting of those points $x \in X$ such that $\Omega^1_{\mathcal{O}_{X,x}/k}$ is not free of rank 2. We first define the *Chow groups* of X as follows.

(i) $CH^0(X)$ is the free Abelian group on the connected components of X.

(ii) $CH^1(X)$ is the Cartier class group $CaCl(X)$ (see [H], Chapter 2, §5), namely the quotient of the group of Cartier divisors modulo the subgroup of principal Cartier divisors. This is known (since X is quasi-projective) to be naturally isomorphic to the group Pic (X) of isomorphism classes of invertible sheaves.
We also have 2 other natural descriptions of $CH^1(X)$. Let $Z^1(X)$ be the group of Cartier divisors D on X such that supp $(D) \cap X_{\rm sing}$ is a finite set. Then the natural map $Z^1(X) \to CH^1(X)$ is surjective, with kernel

$$R^1(X) = \left\{ (f)_X \;\middle|\; \begin{array}{l} f \text{ is an invertible rational function which is} \\ \text{a unit at the generic points of irreducible} \\ \text{components of } X_{\rm sing} \text{ of codimension } 1 \end{array} \right\}.$$

Thus $CH^1(X) \cong Z^1(X)/R^1(X)$.
We may similarly define $Z^1(X)'$ to be the group of Cartier divisors D such that supp (D) meets $X_{\rm sing}$ properly, and let $R^1(X)'$ consist of principal Cartier divisors $(f)_X$, where f is a unit at all generic points of $X_{\rm sing}$. Again, it is easy to see that $Z^1(X)' \to CH^1(X)$ is surjective, with kernel $R^1(X)'$.

(iii) Let $Z^2(X)$ be the free Abelian group on the closed points of $X - X_{\rm sing}$. Let

$$R^2(X) = \text{subgroup of } Z^2(X) \text{ generated by cycles } (i_C)_*((f)_C),$$

where $C \subset X$ is an effective Cartier divisor of pure dimension 1, such that $C \cap X_{\rm sing}$ is finite, and f is an invertible rational function on C, such that f is a unit at each point of $C \cap X_{\rm sing}$. Here $(f)_C$ denotes the 0-cycle associated to the divisor of f on C, and i_C is the inclusion of C into X (which induces a map $(i_C)_*$ on 0-cycles supported on $C - X_{\rm sing}$). Define $CH^2(X) = Z^2(X)/R^2(X)$.

We call purely 1-dimensional effective Cartier divisors C *Cartier curves* on X. We will usually suppress $(i_C)_*$ in the notation. We will use the notation $R(C)$ for the (multiplicative) group of invertible rational functions f which are units on $C \cap X_{\text{sing}}$ (in [BiS], we use instead the notation $R(C,X)$, to emphasize the dependence on X, but this is not so important in this paper).

Remark. We remark that if C is as above, and $C = C_1 + C_2$ where C_i are Cartier divisors on X with no common component, then $R(C) \subset R(C_1) \times R(C_2)$, and for $f \in R(C)$, we have $(f)_C = (f|_{C_1})_{C_1} + (f|_{C_2})_{C_2}$. We can find a unique such decomposition $C = C_1 + C_2$ such that each component of C_1 meets X_{sing}, while $C_2 \cap X_{\text{sing}} = \emptyset$. Then $R(C) = R(C_1) \times R(C_2)$. Further, we may uniquely write $C_2 = \sum_i m_i D_i$ as divisors, where the D_i are distinct (reduced) irreducible curves in X which are disjoint from X_{sing}, and n_i are positive integers. Let $f_i = f|_{D_i}$; then we have an equation between 0-cycles

$$(f|_{C_2})_{C_2} = \sum_i n_i (f_i)_{D_i} = \sum_i (f_i^{n_i})_{D_i}.$$

Hence $R^2(X)$ is generated by cycles $(f)_C$, where we may assume in addition that each irreducible component of C which is disjoint from X_{sing} appears with multiplicity 1 in C.

We now proceed to construct the intersection product, giving the graded ring structure on $CH^*(X) = CH^0(X) \oplus CH^1(X) \oplus CH^2(X)$. There is a natural isomorphism $CH^*(X) = \oplus_{j=1}^s CH^*(Y_j)$ where $Y_1, \dots, Y_s$ are the connected components of X. The endomorphism of $CH^*(X)$ determined by the product with the class of a connected component Y_j of X is defined to be the projection onto the summand $CH^*(Y_j)$. The product $CH^i(X) \otimes CH^j(X) \to CH^*(X)$ must take values in $CH^{i+j}(X)$, *i.e.*, vanishes if $i + j > 2$. Hence the only product which remains to be defined is a pairing

$$CH^1(X) \otimes CH^1(X) \to CH^2(X).$$

The associativity of the product we define will be obvious, and the commutativity will follow from the symmetry of the pairing on $CH^1(X)$.

We define this product in 2 steps. First, suppose D, E are two effective Cartier divisors on X, such that $(D \cup E) \cap X_{\text{sing}}$ is finite, and $D \cap E$ is supported at a finite subset of $X - X_{\text{sing}}$. Then we define

$$D \cdot E = \sum_{P \in D \cap E} I(P; D, E)[P],$$

where $I(P; D, E)$ is the intersection multiplicity at P of D and E, on the surface X (which is non-singular in a neighbourhood of P). Thus, if f, g respectively generate the ideals of D, E in $\mathcal{O}_{P,X}$, then $\mathcal{O}_{P,X}/(f,g)$ has finite length, and this length is defined to be $I(P; D, E)$. Then $D \cdot E$ is a well-defined element of $CH^2(X)$.

Next, for an arbitrary pair of Cartier divisors D, E, choose effective Cartier divisors D_1, D_2, E_1, E_2 such that $D \sim D_1 - D_2$, $E \sim E_1 - E_2$ (here $\sim$ denotes linear equivalence of divisors), and such that (i) $(D_1 \cup D_2 \cup E_1 \cup E_2) \cap X_{\text{sing}}$ is a finite set, and (ii) $D_i \cap E_j$ is a finite subset of $X - X_{\text{sing}}$ for all $1 \le i, j \le 2$. Such a choice of effective Cartier divisors always exists, since X is quasi-projective (one may even choose the divisors to be reduced and very ample). Then define

$$D \cdot E = D_1 \cdot E_1 + D_2 \cdot E_2 - D_1 \cdot E_2 - D_2 \cdot E_1.$$

Here the right side is a well-defined element of $CH^2(X)$, as seen in the first step. Further, if we show that the right side is independent of the choices of the D_i and E_i, then it will also follow that (i) $D \cdot E = E \cdot D$, *i.e.*, the pairing is symmetric, and (ii) $D \cdot E$ depends only on the linear equivalence classes of D and E.

So assume $D \sim D_3 - D_4$, $E \sim E_3 - E_4$ are another pair of such linear equivalences. Then we can find a linear equivalence $D \sim D_5 - D_6$, where D_5, D_6 are also effective Cartier divisors with no common components, $(D_5 \cup D_6) \cap X_{\text{sing}}$ is finite, and such that each of D_5, D_6 intersects each of E_1, E_2, E_3, E_4 in a finite set of smooth points of X. Now note that $D_1 + D_6 \sim D_2 + D_5$, and so there is a rational function f on X with divisor $D_1 + D_6 - D_2 - D_5$; note that f is a unit in $\mathcal{O}_{x,X}$ for any $x \in X - (D_1 \cup D_2 \cup D_5 \cup D_6)$.

Let $\widetilde{E}_j$ be obtained by removing the 0-dimensional irreducible components from E_j (note that these 0-dimensional components lie in X_{sing}). Then $\widetilde{E}_j$ is a Cartier curve in X, for $j = 1, 2$, such that the restriction to $\widetilde{E}_j$ of the rational function f lies in $R(\widetilde{E}_j)$, with divisor

$$(D_1 + D_6) \cdot E_j - (D_2 + D_5) \cdot E_j$$

(here the restriction of f lies in $R(\widetilde{E}_j)$ since $(D_1 \cup D_2 \cup D_5 \cup D_6) \cap E_j$ consists of a finite set of smooth points of X). This implies that

$$(D_1 + D_6) \cdot E_j = (D_2 + D_5) \cdot E_j \in CH^2(X).$$

Hence the expressions $D \sim D_1 - D_2$ and $D \sim D_5 - D_6$ yield the same element in $CH^2(X)$, for the chosen expression $E \sim E_1 - E_2$. We will write this as an equation in $CH^2(X)$

$$(D_1 - D_2) \cdot (E_1 - E_2) = (D_5 - D_6) \cdot (E_1 - E_2).$$

Similar reasoning shows that we also have equations between classes of 0-cycles

$$(D_5 - D_6) \cdot (E_1 - E_2) = (D_5 - D_6) \cdot (E_3 - E_4),$$
$$(D_5 - D_6) \cdot (E_3 - E_4) = (D_3 - D_4) \cdot (E_3 - E_4).$$

Combining the above 3 equations, we have the desired equation

$$(D_1 - D_2) \cdot (E_1 - E_2) = (D_3 - D_4) \cdot (E_3 - E_4).$$

This shows that the intersection product is well-defined on $CH^*(X)$.

3. Another definition of $CH^2(X)$

It is useful to have another description of the relations $R^2(X)$ in the definition of $CH^2(X)$. Let $R^2(X, X_{\text{sing}})$ denote the subgroup of $Z^2(X)$ generated by cycles of the following type.

Let $W \subset X \times \mathbb{P}^1$ be a subscheme of pure codimension 2, and $W_0, W_\infty \in Z^2(X)$, such that

(1) $W \cap (X_{\text{sing}} \times \mathbb{P}^1)$ is a finite set, and $W \cap (X_{\text{sing}} \times \{0, \infty\}) = \emptyset$,
(2) the map $W \longrightarrow \mathbb{P}^1_k$ (obtained by the second projection on $X \times \mathbb{P}^1_k$) is flat over a neighbourhood of $\{0, \infty\}$,
(3) if $p_X : X \times \mathbb{P}^1_k \longrightarrow X$ is the projection, then $W_0 = (p_X)_*(W \cdot X \times \{0\})$, $W_\infty = (p_X)_*(W \cdot X \times \{\infty\})$, where '$\cdot$' denotes the intersection cycle (which makes sense by (1) and (2)), and $(p_X)_*$ denotes the direct image cycle under the proper morphism p_X,
(4) the ideal of W in $X \times \mathbb{P}^1$ at any point $x \in W \cap (X_{\text{sing}} \times \mathbb{P}^1)$ is generated by a regular sequence.

Under the above conditions, $W_0 - W_\infty \in Z^2(X)$. Let $R^2(X, X_{\rm sing})$ be the subgroup generated by such cycles.

PROPOSITION 1

$R^2(X, X_{\rm sing}) = R^2(X)$.

Proof. Clearly $R^2(X) \subset R^2(X, X_{\rm sing})$ by associating to a cycle $(i_C)_*((f)_C)$ the graph $Z \subset C \times \mathbb{P}^1 \subset X \times \mathbb{P}^1$ of the rational map $C \to \mathbb{P}^1$ determined by f.

We now state a useful lemma of Bertini type, whose proof (along standard lines) is left to the reader.

Lemma 1. *Let Z be a reduced locally closed subvariety of a projective space $\mathbb{P}^N$. Then there exists an open dense subset $U \subset \check{\mathbb{P}}^N$ of the dual projective space of hyperplanes in $\mathbb{P}^N$ such that for any hyperplane $H \in U$, the scheme-theoretic intersection $H \cap Z$ is reduced.*

We also need a lemma on intersection multiplicities.

Lemma 2. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension 3, and $I, J \subset R$ ideals such that R/I, R/J are 1-dimensional Cohen–Macaulay, and $I + J$ is $\mathfrak{m}$-primary. Let $t \in R$ be a non zero-divisor on $R/I \cap J$ such that $R/I \cap J + tR$ has finite length. Then (i) t is a non zero-divisor on R/I and R/J (ii) $R/I + tR$, $R/J + tR$ have finite length, and*

$$\text{length}\,(R/I \cap J + tR) = \text{length}\,(R/I + tR) + \text{length}\,(R/J + tR).$$

Proof. Since I, J have height 2, the quotient local rings R/I, R/J each have dimension 1. We have an exact sequence of R-modules

$$0 \to R/I \cap J \to R/I \oplus R/J \to R/I + J \to 0$$

where $R/I + J$ has finite length. Since $R/I + tR$, $R/J + tR$ are quotients of $R/I \cap J + tR$, they have finite length. Since R/I and R/J are 1-dimensional Cohen–Macaulay rings, t must be a non zero-divisor on each of them. Now consider the diagram obtained by multiplication by t on each term of the above short exact sequence; since the third term $R/I + J$ has finite length, multiplication by t has a kernel and cokernel of the same length. From the snake lemma, we then get the desired equality

$$\text{length}\,(R/I \cap J + tR) = \text{length}\,(R/I + tR) + \text{length}\,(R/J + tR). \qquad \square$$

Now suppose $W \subset X \times \mathbb{P}^1$ is a subscheme giving an element of $R^2(X, X_{\rm sing})$. In particular,

(a) W has pure codimension 2 in $X \times \mathbb{P}^1$, and under the second projection, W is flat over a neighbourhood of $\{0, \infty\}$ in $\mathbb{P}^1$,
(b) $W \cap (X_{\rm sing} \times \mathbb{P}^1)$ is a finite set, and $W \cap X_{\rm sing} \times \{0, \infty\} = \emptyset$,
(c) the ideal of W at any point of $W \cap (X_{\rm sing} \times \mathbb{P}^1)$ is generated by a regular sequence (of length 2).

If $p_1 : X \times \mathbb{P}^1 \to X$ is the projection, and if $W_0 = (p_1)_*(W \cdot X \times \{0\})$, $W_\infty = (p_1)_*(W \cdot X \times \{\infty\})$, then $W_0 - W_\infty \in R^2(X, X_{\rm sing})$ is a typical generator of $R^2(X, X_{\rm sing})$. We will show that $W_0 - W_\infty \in R^2(X)$.

We first remark that since $W \to \mathbb{P}^1$ is flat over a neighbourhood of $\{0, \infty\}$, the scheme W is a 1-dimensional Cohen–Macaulay scheme at each point of $W \cdot X \times \{0, \infty\}$. Hence

if we write $W = W' \cup W''$, such that

(a) W', W'' are each Cohen–Macaulay near $W \cdot X \times \{0, \infty\}$, and
(b) W', W'' have no common component, and we have a relation between ideal sheaves $\mathcal{I}_W = \mathcal{I}_{W'} \cap \mathcal{I}_{W''} \subset \mathcal{O}_{X \times \mathbb{P}^1}$,

then by lemma 2, W', W'' are also flat over $\mathbb{P}^1$ in a neighbourhood of $\{0, \infty\}$, and we have equations between 0-cycles

$$[W' \cdot X \times \{0\}] + [W'' \cdot X \times \{0\}] = [W \cdot X \times \{0\}],$$
$$[W' \cdot X \times \{\infty\}] + [W'' \cdot X \times \{\infty\}] = [W \cdot X \times \{\infty\}].$$

We may uniquely write $W = W' \cup W''$ as above such that $W' \cap (X_{\text{sing}} \times \mathbb{P}^1) = \emptyset$, and each irreducible component of W'' meets $X_{\text{sing}} \times \mathbb{P}^1$. Thus we are reduced to separately considering the cases when $W = W'$ and $W = W''$.

In the first case, we may, by a similar decomposition argument, further reduce the proposition to the case when W_{red} is irreducible; if η is its generic point, and $\mathcal{O}_{\eta, W}$ has length r, then for any (closed) point $t \in \mathbb{P}^1$ such that $W \to \mathbb{P}^1$ is flat at t, we have equations between 0-cycles

$$[W \cdot X \times \{t\}] = r[W_{\text{red}} \cdot X \times \{t\}].$$

Hence, we further reduce to the case when W is an integral curve. Then one possibility is that $p_1(W) = C$ is an integral curve in X disjoint from X_{sing}, in which case it is standard that

$$(p_1)_*(W \cdot X \times \{0\}) - (p_1)_*(W \cdot X \times \{\infty\}) = (g)_C,$$

with $g = N_{W/C}(t)$, for the coordinate function t on $\mathbb{P}^1$ (pulled back to W), where $N_{W/C}$ denotes the norm map on rational functions; otherwise, $p_1(W)$ must be a smooth point $x \in X$, in which case $W = \{x\} \times \mathbb{P}^1$, and $(p_1)_*(W \cdot X \times \{0\}) - (p_1)_*(W \cdot X \times \{\infty\})$ is 0 as a cycle, and there is nothing to prove.

So we are left with considering the second case, when $W = W''$, i.e., we may assume also that each irreducible component of W meets $X_{\text{sing}} \times \mathbb{P}^1$. This case is the heart of the proof of proposition 1.

We further reduce at once to the case when X is also connected. Now fix a very ample invertible sheaf $\mathcal{O}_{X \times \mathbb{P}^1}(1)$, giving a (locally closed) projective embedding. Let $\mathcal{I}_W$ denote the ideal sheaf of W in $X \times \mathbb{P}^1$. Let $m_1 < m_2$ be sufficiently large positive integers, and let $g_j \in H^0(X \times \mathbb{P}^1, \mathcal{I}_W(m_j))$ be sufficiently general sections, for $j = 1, 2$ (first choose a general section g_1, then choose g_2 depending on the choice of g_1). Let H be the zero-scheme of g_1 on $X \times \mathbb{P}^1$, and let E be the scheme-theoretic closure in $X \times \mathbb{P}^1$ of the zero-scheme of g_2 on $H - W$.

Lemma 3. *For sufficiently large $m_1 < m_2$, if g_1, g_2 are general, then H, E have the following properties.*

(i) H and E are both reduced; H is purely of codimension 1, *and E is purely of codimension* 2.
(ii) g_1, g_2 form a regular sequence of forms on $X \times \mathbb{P}^1$.
(iii) $E \cap (X_{\text{sing}} \times \{0, \infty\}) = \emptyset$, and $E \cap W \cap (X_{\text{sing}} \times \mathbb{P}^1) = \emptyset$.
(iv) If $p_1 : X \times \mathbb{P}^1 \to X$ is the first projection, then (a) $p_1 : H \to p_1(H)$ is finite over a neighbourhood of $p_1(H) \cap X_{\text{sing}}$, and (b) $E \cap (X_{\text{sing}} \times \mathbb{P}^1) \to p_1(E) \cap X_{\text{sing}}$ is bijective.

(v) $p_1 : H \to X$ *is étale at each point of* $E \cap (X_{\text{sing}} \times \mathbb{P}^1)$.
(vi) E *is a Cartier divisor in* H *in a neighbourhood of* $E \cap X_{\text{sing}} \times \mathbb{P}^1$.
(vii) If D *is the Cartier divisor on* H *defined by the vanishing of* $g_2|_H$, *then we have equations between 0-cycles*

$$[D \cdot X \times \{0\}] = [W \cdot X \times \{0\}] + [E \cdot X \times \{0\}],$$
$$[D \cdot X \times \{\infty\}] = [W \cdot X \times \{\infty\}] + [E \cdot X \times \{\infty\}].$$

Proof. First, we claim that g_1, g_2 form a regular sequence of forms. Now g_1 is a non zero-divisor, since it may be chosen so as not to vanish at any generic point of $X \times \mathbb{P}^1$ (none of which lies in W).

We claim $H \cap W$ does not contain any associated point of H. Suppose, to the contrary, that $x \in H \cap W$ is an associated point of H. Then the maximal ideal of $\mathcal{O}_{x,X\times\mathbb{P}^1}$ is associated to the principal ideal defining H; hence this local ring has depth 1. Since W has pure codimension 2, this local ring has dimension ≥ 2. If it is regular, then it has depth ≥ 2; if it is non-regular, so that $x \in W \cap X_{\text{sing}} \times \mathbb{P}^1$, then the ideal of W in $\mathcal{O}_{x,X\times\mathbb{P}^1}$ is generated by a regular sequence of length 2, so the local ring has depth ≥ 2. In either case, we have a contradiction; this proves the claim.

Now for a general choice of g_2, its zero-scheme avoids the finite set of associated points of H, and so g_2 is a non zero-divisor on $H = \{g_1 = 0\}$, *i.e.*, g_1, g_2 is a regular sequence of forms.

Next, we note that $H^0(X \times \mathbb{P}^1, \mathcal{I}_W(m))$ gives a projective embedding of $(X \times \mathbb{P}^1) - W$, for any sufficiently large m. Hence from the Bertini-type lemma 1, we see that $H - W$ is reduced for a general choice of g_1. The ideal defining H in a local ring $\mathcal{O}_{x,X\times\mathbb{P}^1}$ is unmixed, if $x \in (X - X_{\text{sing}}) \times \mathbb{P}^1$, since the local ring is then regular; also, its minimal primary components are prime, since all generic points of H lie in $H - W$. Hence the ideal of H in such local rings is radical. Thus $H - (W \cap X_{\text{sing}} \times \mathbb{P}^1)$ is reduced, *i.e.*, H is reduced in the complement of a finite subset. Finally, if the ideal of H in $\mathcal{O}_{x,X\times\mathbb{P}^1}$, for $x \in W \cap X_{\text{sing}} \times \mathbb{P}^1$, is not radical, then the maximal ideal must be an associated prime of the (principal) ideal of H, contradicting that the depth of $\mathcal{O}_{x,X\times\mathbb{P}^1}$ is ≥ 2 (since, as noted before, the ideal of W in $\mathcal{O}_{x,X\times\mathbb{P}^1}$ is generated by a regular sequence of length 2). Hence the ideal of H in any local ring $\mathcal{O}_{x,X\times\mathbb{P}^1}$ is radical, *i.e.*, H is reduced.

Now again by the Bertini-type lemma, $E - W$ is reduced, and is purely of codimension 1 in $H - W$. Since E is the scheme theoretic closure of $E - W$, we note that E is reduced, and purely of codimension 2.

Observe that since $W \cap X_{\text{sing}} \times \{0, \infty\} = \emptyset$, and $\dim X_{\text{sing}} \times \{0, \infty\} = \dim X_{\text{sing}} \leq 1$, we have that for general g_1, the divisor H has finite intersection with $X_{\text{sing}} \times \{0, \infty\}$, and $E \cap (X_{\text{sing}} \times \{0, \infty\}) = \emptyset$.

For $x \in W \cap X_{\text{sing}} \times \mathbb{P}^1$, the ideal $\mathcal{I}_{W,x} \subset \mathcal{O}_{x,X\times\mathbb{P}^1}$ is given to be generated by a regular sequence of length 2. Hence, by Nakayama's lemma, for a general choice of g_1, g_2 we may assume that, with respect to a suitable trivialization of $\mathcal{O}_{X\times\mathbb{P}^1}(1)$ at each such point x, the images of g_1, g_2 generate the ideal $\mathcal{I}_{W,x}$. This means that the subscheme defined by the vanishing of g_1 and g_2 coincides with W in a neighbourhood of $W \cap (X_{\text{sing}} \times \mathbb{P}^1)$. Equivalently, $E \cap W \cap (X_{\text{sing}} \times \mathbb{P}^1) = \emptyset$.

We have thus proved (i), (ii) and (iii).

Now we prove (iv). Let r be the positive integer such that

$$\mathcal{O}_{X\times\mathbb{P}^1}(1)\,|_{\{x\}\times\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(r),$$

for all $x \in X$ (recall X is connected). For sufficiently large m_1, we then have that the restriction map,

$$H^0(X \times \mathbb{P}^1, \mathcal{I}_W(m_1)) \to H^0(\{x\} \times \mathbb{P}^1, \mathcal{I}_{W \cap \{x\} \times \mathbb{P}^1}(rm_1)),$$

is surjective, for all $x \in X$. Further, the dimension of

$$H^0(\{x\} \times \mathbb{P}^1, \mathcal{I}_{W \cap \{x\} \times \mathbb{P}^1}(rm_1)),$$

may be assumed to be at least 3, for each $x \in X$ such that $\{x\} \times \mathbb{P}^1 \not\subset W$. Let $U \subset X$ be the open subset of such points x. Then $X_{\mathrm{sing}} \subset U$, by hypothesis. A dimension count now implies that for a general choice of g_1, we have that $\{x\} \times \mathbb{P}^1 \not\subset H$ for all $x \in U$ (the sections for which this does not hold are the union of a family of subspaces of codimension ≥ 3 parametrized by points of U, *i.e.*, these sections form a subset of codimension ≥ 1 in all sections). Thus $p_1 : H \cap U \times \mathbb{P}^1 \to U$ is a proper morphism with finite fibres, hence a finite morphism. This proves (iv) (a).

Next, $H \cap (X_{\mathrm{sing}} \times \mathbb{P}^1)$ has dimension at most that of X_{sing}, *i.e.*, ≤ 1. We have chosen m_2 to be large enough so that $H^0(X \times \mathbb{P}^1, \mathcal{I}_W(m_2))$ embeds $(X \times \mathbb{P}^1) - W$ in a projective space. We claim that for a general choice of g_2, the intersection,

$$(H - W) \cap \{g_2 = 0\} \cap (X_{\mathrm{sing}} \times \mathbb{P}^1) = E \cap (X_{\mathrm{sing}} \times \mathbb{P}^1),$$

maps injectively under the projection to X; equivalently, the general hyperplane section of $(H - W) \cap (X_{\mathrm{sing}} \times \mathbb{P}^1)$ maps injectively to its image under p_1. This is again proved by a dimension count: the hyperplanes passing through 2 points of a given fibre of $H - W \to X$ form a subset of codimension 2 in all hyperplanes; since the locus of pairs of points of $(H - W) \cap (X_{\mathrm{sing}} \times \mathbb{P}^1)$ in such fibres is at most 1-dimensional, the general hyperplane does not pass through two such points in any fibre. This gives (iv) (b).

Now $H \to X$ may be assumed to be separable, since g_1 is general. Hence for some non-empty open subset $V \subset U$, the map $H \cap (V \times \mathbb{P}^1) \to V$ is finite and étale. Further, for any finite set T of points $x \in U$ such that $W \cap \{x\} \times \mathbb{P}^1$ is reduced (or empty), we can choose H so that $T \subset V$. In particular, since $X_{\mathrm{sing}} \subset U$, while $W \cap (X_{\mathrm{sing}} \times \mathbb{P}^1)$ is finite, we can arrange that V meets each curve in X_{sing}, and each isolated point $x \in X_{\mathrm{sing}}$ with $\{x\} \times \mathbb{P}^1 \cap W = \emptyset$. Now $X_{\mathrm{sing}} - V$ is a finite subset of U, with a finite preimage in H; for a general choice of g_2, we can thus arrange that g_2 does not vanish at any point of $(H - W) \cap ((X_{\mathrm{sing}} - V) \times \mathbb{P}^1)$. This means that $E \cap ((X_{\mathrm{sing}} - V) \times \mathbb{P}^1) = \emptyset$, and so $H \to X$ is étale at all points of $E \cap (X_{\mathrm{sing}} \times \mathbb{P}^1)$. This gives (v).

The property (vi) is clear, since $E - W \subset H - W$ is a Cartier divisor, and $E \cap W \cap (X_{\mathrm{sing}} \times \mathbb{P}^1) = \emptyset$. To prove (vii), note that we need only prove a local assertion about intersection multiplicities at each point $x \in E \cap W \cap (X \times \{0, \infty\})$. Now D is Cohen–Macaulay at such points x, since $x \in (X - X_{\mathrm{sing}}) \times \mathbb{P}^1$ is a smooth point. E and W have no common irreducible component, since each irreducible component of W meets $X_{\mathrm{sing}} \times \mathbb{P}^1$, and D agrees with W in a neighbourhood of $W \cap (X_{\mathrm{sing}} \times \mathbb{P}^1)$. Also, $E \cup W = D$ set-theoretically; D agrees with W in a neighbourhood of each generic point of W (since any generic point of W specializes to a point of $X_{\mathrm{sing}} \times \mathbb{P}^1$), and D agrees with E at each generic point of E. Hence $E \cup W$, defined by the intersection of ideal sheaves in $X \times \mathbb{P}^1$, is a 1-dimensional subscheme of D which coincides with it at all generic points. Hence $E \cup W$ coincides with D on the (open dense) subset of Cohen–Macaulay points of D, which includes the points lying over $\{0, \infty\} \subset \mathbb{P}^1$. Now (vii) follows easily, using lemma 2. □

As a consequence of this lemma, we note that for general choices of the forms g_1, g_2, the resulting subscheme E is such that $E \to p_1(E)$ has singleton fibres over the finite set of points of $p_1(E) \cap X_{\mathrm{sing}}$, and $H \to X$ is étale at these points. Since E is a Cartier divisor on H at these points, it follows that $p_1(E)$ is a Cartier divisor on X near points of $X_{\mathrm{sing}} \cap p_1(E)$. In particular, $p_1(E)$ is a (reduced) Cartier curve in X. If t is the standard (coordinate) function on $\mathbb{P}^1$ with divisor $(0) - (\infty)$, then $f = N_{E/p_1(E)}(p_2^*(t)\,|_E)$ lies in $R(p_1(E))$ (here, $N_{E/p_1(E)}$ denotes the norm map on rational functions). Hence the zero cycle,

$$(p_1)_*(E \cdot X \times \{0\}) - (p_1)_*(E \cdot X \times \{\infty\} = (f)_{p_1(E)},$$

lies in $R^2(X)$.

Now choose a general $g_3 \in H^0(X \times \mathbb{P}^1, \mathcal{O}_{X \times \mathbb{P}^1}(m_2))$. Let W' be the subscheme of $X \times \mathbb{P}^1$ defined by $g_1 = g_3 = 0$. Then W' is a reduced Cartier divisor on H. As in the proof of the above lemma, we may further assume that (a) $W' \cap X_{\mathrm{sing}} \times \{0, \infty\} = \emptyset$ (since $H \cap X_{\mathrm{sing}} \times \{0, \infty\}$ may be assumed to be finite), (b) $W' \to p_1(W')$ is injective on $W' \cap (X_{\mathrm{sing}} \times \mathbb{P}^1)$ (c) $H \to X$ is étale at points of $W' \cap (X_{\mathrm{sing}} \times \mathbb{P}^1)$.

Let $f' = N_{W'/p_1(W')}(p_2^*(t)|_{W'})$. Then, as above, we see that $p_1(W')$ is a Cartier curve in X, and $f' \in R(p_1(W'))$. Hence,

$$(p_1)_*(W' \cdot X \times \{0\}) - (p_1)_*(W' \cdot X \times \{\infty\} = (f')_{p_1(W')},$$

also lies in $R^2(X)$.

Let

$$h_0 = (g_2/g_3)|_{H \cap (X \times \{0\})},$$
$$h_\infty = (g_2/g_3)|_{H \cap (X \times \{\infty\})}.$$

Note that

$$H \cap (X \times \{0\}) \cong p_1(H \cap (X \times \{0\})) \subset X,$$

is a Cartier curve, say H_0, and $h_0 \in R(H_0)$; similarly,

$$H \cap (X \times \{\infty\}) \cong p_1(H \cap (X \times \{\infty\})) = H_\infty \subset X,$$

is also a Cartier curve, and $h_\infty \in R(H_\infty)$. Finally, we have an equation between 0-cycles (using (vii) of lemma 3)

$$\begin{aligned}(p_1)_*(W \cdot X \times \{0\} - W \cdot X \times \{\infty\}) &= (h_0)_{H_0} - (h_\infty)_{H_\infty} \\ &\quad + (p_1)_*(W' \cdot X \times \{0\} - W' \cdot X \times \{\infty\} - E \cdot X \times \{0\} + E \cdot X \times \{\infty\}) \\ &= (h_0)_{H_0} - (h_\infty)_{H_\infty} + (f')_{p_1(W')} - (f)_{p_1(E)}.\end{aligned}$$

All the terms in the third expression lie in $R^2(X)$, and hence so does the first expression. This completes the proof of the proposition. □

4. Pulling back cycle classes under morphisms to homogeneous varieties

Our aim in this section is to construct a pull-back homomorphism of graded rings $f^*: CH^*(H) \to CH^*(X)$ associated to a morphism $f: X \to H$, where H is a homogeneous space for an algebraic group. We will do this for suitable homogeneous spaces H, and suitable morphisms.

In the next lemma, recall that two closed subschemes Y_1, Y_2 in a variety Z *intersect properly* if for any irreducible components Y_1', Y_2' of Y_1, Y_2 respectively, $Y_1' \cap Y_2'$ is of codimension $\geq \operatorname{codim}_Z Y_1' + \operatorname{codim}_Z Y_2'$ in Z.

Lemma 4. *Let G be a linear algebraic group, H a homogeneous space for G. Let Z be a reduced k-variety, $f: Z \to H$ a morphism, and Y an irreducible subvariety of H of codimension i. Then there is a non-empty open subset $U(f, Y) \subset G$ such that for each $g \in U(f, Y)$, the following properties hold.*

(i) *The scheme-theoretic inverse image $f^{-1}(gY)$ is purely of codimension i.*

(ii) *The following intersections are proper: (a) gY_{sing} and $f(Z)$ (in H), (b) $f^{-1}(gY)$ and Z_{sing} (in Z) (c) $f^{-1}(gY_{\text{sing}})$ and Z_{sing} (in Z).*

(iii) *$f^{-1}(gY)$ is a local complete intersection in Z at all generic points of $f^{-1}(gY) \cap Z_{\text{sing}}$.*

(iv) *The sheaves $\mathcal{T}or_i^{f^{-1}\mathcal{O}_H}(f^{-1}\mathcal{O}_{gY}, \mathcal{O}_Z)$ vanish for all $i > 0$.*

Proof. We may write $H = G/P$, where P is the isotropy subgroup-scheme of G at a chosen base-point $h_0 \in H$. Let $\Gamma \subset G \times Z$ be the incidence subscheme

$$\Gamma = \{(g, x) \in G \times Z | f(x) \in gY\},$$

with projections $p: \Gamma \to G$, $q: \Gamma \to Z$. Then $p^{-1}(g) = f^{-1}(gY)$ is the scheme-theoretic inverse image of gY under $f: X \to H$.

Consider the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\alpha} & Y \times Z \\ \gamma \downarrow & & \downarrow \beta \\ G \times H & \xrightarrow{\delta} & H \times H \end{array}$$

where $\alpha(g, x) = (g^{-1}f(x), x)$, $\beta(h, x) = (h, f(x))$, $\gamma(g, x) = (g, f(x))$ and $\delta(g, h) = (g^{-1}h, h)$. We claim that (i) this is a fibre product diagram, and (ii) δ is flat, with all fibres isomorphic to P. (i) is easily proved, using that $Y \to H$ is an inclusion. It is clear that $\delta^{-1}(h_0, h_0) = P \times \{h_0\}$. Further, $G \times G$ acts on $G \times H$ by $(g_1, g_2) \cdot (g, h) = (g_2 g g_1^{-1}, g_2 h)$, and also on $H \times H$, by $(g_1, g_2) \cdot (h', h'') = (g_1 h', g_2 h'')$. One verifies at once that δ is $G \times G$-equivariant, and $G \times G$ acts transitively on $H \times H$. Hence all fibres of δ are isomorphic; since δ is flat over a dense open subset of $H \times H$, it is flat over all $G \times G$-translates of this open set, *i.e.*, δ is flat.

In particular, if Z is irreducible, then Γ is purely of dimension $\dim Z + \dim Y + \dim P$, *i.e.*, of codimension i in $G \times Z$. Hence for an open dense subset $U \subset G$, the fibre $f^{-1}(gY)$ is either empty, or purely of codimension i in Z.

If Z is reducible, we may apply the same argument to each irreducible component; we may also apply it replacing Z by an irreducible component of Z_{sing} and/or replacing Y by an irreducible component of Y_{sing}. This proves (i) and (ii) of the lemma. In particular, note that for general $g \in G$, the generic points of $f^{-1}(gY) \cap Z_{\text{sing}}$ lie over *smooth* points of gY.

Now $\pi: G \times Z \to H$, $(g, x) \mapsto g^{-1}f(x)$, is surjective and $\Gamma = \pi^{-1}(Y)$ is the scheme-theoretic inverse image. Let G act on $G \times Z$ by $g \cdot (g', x) = (g'g^{-1}, x)$; then π is G-equivariant. By generic flatness, π is flat over a non-empty (dense) open subset of H; since π is G-equivariant, it is also flat over all G-translates of this open subset, *i.e.*, π is flat.

By generic flatness, the morphism $p: \Gamma \to G$ is also flat over an open dense subset of G. Let $\mathcal{E}^\bullet \to \mathcal{O}_Y \to 0$ be a resolution of $\mathcal{O}_Y$ by locally free $\mathcal{O}_H$-modules. Then

$\pi^*\mathcal{E}^\bullet \to \mathcal{O}_\Gamma \to 0$ is a locally free resolution of $\mathcal{O}_\Gamma$ as an $\mathcal{O}_{G\times Z}$-module. If p is flat over $g \in G$, and $i_g : Z \to G \times Z$, $x \mapsto (g,x)$ is the inclusion, then

$$i_g^*\pi^*\mathcal{E}^\bullet \to i_g^*\mathcal{O}_\Gamma \to 0$$

is also a resolution. Hence $\mathcal{T}or_i^{(\pi\circ i_g)^{-1}\mathcal{O}_H}((\pi\circ i_g)^{-1}\mathcal{O}_Y, \mathcal{O}_Z) = 0$ for all $i > 0$. Since $\pi \circ i_g \colon Z \to H$ is given by $x \mapsto g^{-1}f(x)$, we have that $\pi \circ i_g = T_{g^{-1}} \circ f$, where $T_{g^{-1}}$: $H \to H$ is translation by g^{-1}, which is an isomorphism. Hence $\mathcal{T}or_i^{f^{-1}\mathcal{O}_H}(f^{-1}T_{g^{-1}}^{-1}\mathcal{O}_Y, \mathcal{O}_Z) = 0$ for all $i > 0$. But $T_{g^{-1}}^{-1}(Y) = gY$, and so $T_{g^{-1}}^{-1}\mathcal{O}_Y = \mathcal{O}_{gY}$. Thus the sheaves $\mathcal{T}or_i^{f^{-1}\mathcal{O}_H}(f^{-1}\ \mathcal{O}_{gY}, \mathcal{O}_Z)$ vanish for all $i > 0$, for all such $g \in G$.

In particular, since $Y - Y_{\text{sing}}$ is a local complete intersection in H, we see that for general $g \in G$, the (locally closed) subscheme $f^{-1}(g(Y - Y_{\text{sing}}))$ is a local complete intersection in Z. But $f^{-1}(g(Y - Y_{\text{sing}}))$ contains all generic points of $f^{-1}(gY) \cap Z_{\text{sing}}$. □

Remark. In the sequel, we will abuse notation and write $\mathcal{T}or_i^{\mathcal{O}_H}(\mathcal{O}_{gY}, \mathcal{O}_X)$ to mean $\mathcal{T}or_i^{f^{-1}\mathcal{O}_H}(f^{-1}\mathcal{O}_{gY}, \mathcal{O}_Z)$. We will also express the condition that $\mathcal{T}or_i^{\mathcal{O}_H}(\mathcal{O}_{gY}, \mathcal{O}_X) = 0$ for all $i > 0$ by the statement that $\mathcal{O}_X$ and $\mathcal{O}_{gY}$ are *Tor-independent* over $\mathcal{O}_H$.

COROLLARY 1

Let $f : X \to H$ be a morphism to a G-homogeneous space, as above, and Y a subscheme of H of codimension 2, with (reduced) irreducible components $Y_1, \ldots, Y_r$ of codimension 2, and associated cycle $[Y] = \sum_{j=1}^r n_j[Y_j]$. Then there is an open dense subset $W \subset G$ such that for all $g \in W$, we have that $f^{-1}(gY)$ is a 0-dimensional subscheme of $X - X_{\text{sing}}$, and we have an equality between 0-cycles $[f^{-1}(gY)] = \sum_{j=1}^r n_j[f^{-1}(gY_j)]$.

Proof. Let $y_j \in Y_j$ be the generic point, for each j. There is an affine open neighbourhood $U_j = \operatorname{Spec} A_j$ of y_j, for each j, such that (i) $Y_k \cap U_j = \emptyset$ for $k \neq j$, and (ii) if $P_j \subset A_j$ is the prime ideal defining Y_j, and $Y \cap U_j = \operatorname{Spec} B_j$, then B_j has a finite filtration by A_j-submodules with all quotients isomorphic to A_j/P_j; the number of proper terms in the filtration of B_j is necessarily equal to n_j.

Consider $Y - \cup_{j=1}^r U_j$, which is a proper closed subset of Y, and is of codimension ≥ 3 in H. Hence by lemma 4, for an open subset of elements $g \in G$, we have $f(X) \cap gY \subset \cup_{j=1}^r g(Y \cap U_j)$. We may further assume that for each j, the inverse image $f^{-1}(gY_j)$ is a 0-dimensional subscheme supported on $X - X_{\text{sing}}$, and $\mathcal{O}_X$ and $\mathcal{O}_{gY_j}$ are Tor-independent. This means that if $x \in f^{-1}(gY)$, then $f(x) = y \in Y_k \cap U_k$ for some k. Then in the 0-cycle $[f^{-1}(gY)]$, the point x has a coefficient equal to the length of $\mathcal{O}_{X,x} \otimes_{A_k} B_k$, while $\sum_{j=1}^r n_j[f^{-1}(gY_j)]$ has a coefficient n_j (length of $\mathcal{O}_{X,x} \otimes_{A_j} A_j/P_j$). But $\mathcal{O}_{X,x}$ and A_j/P_j are Tor-independent over A_j, and B_j has a filtration with n_j quotients each isomorphic to A_j/P_j. Hence $\mathcal{O}_{X,x} \otimes_{A_k} B_k$ has a filtration with n_j quotients all isomorphic to $\mathcal{O}_{X,x} \otimes_{A_j} A_j/P_j$. This means that x has the same coefficient in $[f^{-1}(gY)]$ and $\sum_j n_j[f^{-1}(gY_j)]$. □

We next prove a simple lemma on flatness.

Lemma 5. Let A be a discrete valuation ring, B, C flat A-algebras and $\varphi : B \to C$ a homomorphism of A-algebras. Let M be a B-module such that

(i) M is A-flat;
(ii) if $t \in A$ is a local parameter, then $Tor_i^{B/tB}(C/tC, M/tM) = 0$ for all $i > 0$.

Then $M \otimes_B C$ is A-flat.

Proof. Let $P^\bullet \longrightarrow M \longrightarrow 0$ be a B-projective resolution. Since this is an exact sequence of flat A-modules, $P^\bullet \otimes_B B/tB \longrightarrow M/tM \longrightarrow 0$ is a resolution of M/tM by projective B/tB-modules (where we have used that for a B-module N, we have natural isomorphisms $N \otimes_A A/tA \cong N \otimes_B B/tB \cong N/tN$). Since $\mathrm{Tor}_i^{B/tB}(M/tM, C/tC) = 0$ for all $i > 0$, we deduce that $P^\bullet \otimes_B C/tC \longrightarrow M \otimes_B C/tC \longrightarrow 0$ is a C/tC-projective resolution of $M \otimes_B C/tC$.

We have a short exact sequence of complexes

$$0 \longrightarrow P^\bullet \otimes_B C \xrightarrow{\cdot t} P^\bullet \otimes_B C \longrightarrow P^\bullet \otimes_B C/tC \longrightarrow 0.$$

Since the third complex has vanishing H_1, we have a corresponding short exact sequence of H_0 modules

$$0 \longrightarrow M \otimes_B C \xrightarrow{\cdot t} M \otimes_B C \longrightarrow M \otimes_B C/tC \longrightarrow 0.$$

This means precisely that $M \otimes_B C$ is A-flat, since A is a discrete valuation ring with parameter t. □

Lemma 6. *Let G be an algebraic group, H a G-homogeneous space, X a reduced k-scheme of finite type, and $f : X \longrightarrow H$ a morphism. Let G act on $H \times \mathbb{P}^1$ by the trivial action on $\mathbb{P}^1$ (so that $g \cdot (y, u) = (gy, u)$). Let $W \subset H \times \mathbb{P}^1$ be a subscheme of pure codimension i, flat over a neighbourhood of $\{0, \infty\}$ in $\mathbb{P}^1$. For $g \in G$, let*

$$W(g) = (f \times 1_{\mathbb{P}^1})^{-1}(gW) \subset X \times \mathbb{P}^1.$$

Then there exists an open dense subset $U \subset G$ such that for all $g \in U$, the following properties hold.

(a) $W(g)$ is purely of codimension i in $X \times \mathbb{P}^1$.
(b) $W(g)$ intersects $X_{\mathrm{sing}} \times \mathbb{P}^1$ and $X_{\mathrm{sing}} \times \{0, \infty\}$ properly.
(c) $W(g)$ is flat over (a neighbourhood of) $\{0, \infty\} \subset \mathbb{P}^1$.
(d) $W(g)$ is a local complete intersection in $X \times \mathbb{P}^1$ at each generic point of $W(g) \cap (X_{\mathrm{sing}} \times \mathbb{P}^1)$.

Proof. Let $G' = G \times \mathrm{Aut}(\mathbb{P}^1)$. Then $H \times \mathbb{P}^1$ is a G'-homogeneous space. If $g' = (g, \tau) \in G \times \mathrm{Aut}(\mathbb{P}^1)$, then

$$(f \times 1_{\mathbb{P}^1})^{-1}(g'W) = \tau (f \times 1_{\mathbb{P}^1})^{-1}(gW) = \tau W(g).$$

Now $W(g)$ is purely of codimension i, intersects $X_{\mathrm{sing}} \times \mathbb{P}^1$ properly, and is a local complete intersection in $X \times \mathbb{P}^1$ at the generic points of $W(g) \cap (X_{\mathrm{sing}} \times \mathbb{P}^1)$, if and only if $\tau W(g)$ has the analogous properties. By lemma 4, it follows that for all g in some non-empty open subset $U' \subset G$, the subscheme $W(g) \subset X \times \mathbb{P}^1$ satisfies (a) and (d), and intersects $X_{\mathrm{sing}} \times \mathbb{P}^1$ properly.

Apply lemma 4 again to each irreducible component of $W \cap H \times \{0, \infty\}$, and to the morphisms $X \longrightarrow H \cong H \times \{0\}$, $X \longrightarrow H \cong H \times \{\infty\}$. We obtain a non-empty open subset $U'' \subset G$ such that for $g \in U''$, $W(g) \cap X \times \{0, \infty\}$ intersects $X_{\mathrm{sing}} \times \{0, \infty\}$ properly. This proves (b).

Now W (and hence any translate gW) is flat over $\mathbb{P}^1$ in a neighbourhood of $\{0, \infty\}$. By lemma 4, we may also assume that for all $g \in U''$, the sheaves $\mathcal{T}or_i^{\mathcal{O}_H}(\mathcal{O}_X, \mathcal{O}_{gW \cap H \times \{t\}})$ vanish for all $i > 0$, and $t = 0, \infty$. Hence by lemma 5, we deduce that $\mathcal{O}_{W(g)}$ is flat over $\mathbb{P}^1$ at $0, \infty$ (take B, C, M of lemma 5 to be suitable local rings of $H \times \mathbb{P}^1$, $X \times \mathbb{P}^1$ and gW, respectively; A is the local ring of $\mathbb{P}^1$ at either 0 or ∞). This proves (c). □

Lemma 7. Let H be a homogeneous space for a connected linear algebraic group G, and Y an irreducible subvariety of H of codimension 2. Let $f : X \to H$ be a morphism. Then there is a dense Zariski open subset $V \subset G$ such that for any $g_1, g_2 \in V$, the subschemes $f^{-1}(g_1 Y), f^{-1}(g_2 Y)$ are both of pure codimension 2 in $X - X_{\text{sing}}$, and determine the same element of $CH^2(X)$.

Proof. We first show that there exists an irreducible variety B, a dominant morphism $v : B \times G \to G \times G$ and a reduced, irreducible subvariety $\Sigma \subset B \times G \times H \times \mathbb{P}^1$ such that

(i) $\Sigma \to B \times G$ and $\Sigma \to \mathbb{P}^1$ are flat,
(ii) for $(w, g) \in B \times G$, the scheme-theoretic fibre $\Sigma_{(w,g)} = \Sigma \otimes_{\mathcal{O}_{B\times G}} k(w, g)$ is reduced and irreducible, of pure codimension 2 on $H \times \mathbb{P}^1$,
(iii) if $v(w, g) = (g_1, g_2) \in G \times G$, then $\Sigma_{(w,g)} \otimes_{\mathcal{O}_{\mathbb{P}^1}} k(0) = g_1 Y$, $\Sigma_{(w,g)} \otimes_{\mathcal{O}_{\mathbb{P}^1}} k(\infty) = g_2 Y$,
(iv) Σ is mapped into itself under the action of G on $B \times G \times H \times \mathbb{P}^1$ given by $g'(w, g, h, t) = (w, g'g, g'h, t)$; further, $g'\Sigma_{(w,g)} = \Sigma_{(w,g'g)}$.

Since G is a linear algebraic group, it is unirational. Let $\overline{G}$ be a projective closure of G. Then there is an irreducible variety B, a morphism $p : B \times \mathbb{P}^1 \to \overline{G}$, and a non-empty open subset $U_1 \subset G$, such that for any two points $g_1, g_2 \in U_1$, there exists $w \in B$ such that $p(w, 0) = g_1, p(w, \infty) = g_2$. We may assume (after shrinking B) that $p(B \times \{0, \infty\}) \subset G$. Let

$$\Sigma = \text{closure in } B \times G \times H \times \mathbb{P}^1 \text{ of}$$
$$\Sigma^0 = \{(w, g, h, t) \in B \times G \times H \times \mathbb{P}^1 \mid p(w, t) \in G,\ h \in g \cdot p(w, t) \cdot Y\}.$$

Consider the (projection) morphism $\pi : \Sigma \to G \times B \times \mathbb{P}^1$, $(w, g, h, t) \mapsto (g, w, t)$. There is an open (dense) subset $p^{-1}(G) \subset B \times \mathbb{P}^1$, and hence an open subset $G \times p^{-1}(G) \subset G \times B \times \mathbb{P}^1$; by construction, $G \times p^{-1}(G)$ is the image of Σ^0, and Σ^0 is closed in $\pi^{-1}(G \times p^{-1}(G))$. Further, if $(w, t) \in p^{-1}(G)$, then the fibre $\pi^{-1}(g, w, t) = \{(g, w)\} \times g \cdot p(w, t) \cdot Y \times \{t\} \cong g \cdot p(w, t) \cdot Y$. In particular, $\pi : \Sigma^0 \to G \times p^{-1}(G)$ is surjective with reduced, irreducible fibres. We claim further that π is flat. Indeed, there is a morphism $\varphi : G \times p^{-1}(G) \to G$, $(g, w, t) \mapsto g \cdot p(w, t)$; if $\Gamma = \{(g, h) \in G \times H | g^{-1}h \in Y\}$, then there is a fibre product diagram

$$\begin{array}{ccc} \Sigma^0 & \longrightarrow & \Gamma \\ \pi \downarrow & & \downarrow \psi \\ G \times p^{-1}(G) & \xrightarrow{\varphi} & G \end{array}$$

where $\psi : \Gamma \to G$ is the projection. So the flatness of π follows from that of ψ, which holds because ψ is generically flat and G-equivariant (where G acts on Γ by $g' \cdot (g, h) = (g'g, g'h)$). Hence Σ^0 is irreducible, and the projections $\theta : \Sigma^0 \to B \times G$, $(w, g, h, t) \mapsto (w, g)$, and $\Sigma^0 \to \mathbb{P}^1$, $(w, g, h, t) \mapsto t$, are also flat. Further, for each fixed $(w, g) \in B \times G$, the map $\Sigma^0_{(w,g)} = \theta^{-1}(w, g) \to \mathbb{P}^1$ is flat.

We deduce first that Σ is reduced and irreducible, and for any $(w, t) \in p^{-1}(G)$, we have $\Sigma \otimes_{\mathcal{O}_{B\times G\times\mathbb{P}^1}} k(w, g, t) = g \circ p(w, t)Y$. In particular, $\Sigma \to B \times G \times \mathbb{P}^1$ is separable, with generically irreducible fibres. Hence $\Sigma \to B \times G$ (extending θ) is also separable with generically irreducible fibres. The map $\Sigma \to B \times G$ is G-equivariant, for the G-actions:

$$g'(w, g, h, t) = (w, g'g, g'h, t), \quad g'(w, g) = (w, g'g).$$

After shrinking B, we can thus arrange that Σ is flat over $B \times G$, and $\Sigma_{(w,g)} = \Sigma \otimes_{\mathcal{O}_{B\times G}} k(w, g)$ is reduced and irreducible, and $\Sigma_{(w,g)} \otimes_{\mathcal{O}_{\mathbb{P}^1}} k(t) = g \cdot p(w, t)Y$ if $p(w, t) \in G$.

Note further that since $\Sigma \to \mathbb{P}^1$ is flat over the open dense subscheme Σ^0, $\Sigma \to \mathbb{P}^1$ is flat. Finally, let $v : B \times G \to G \times G$ be the morphism $v(w, g) = (g \cdot p(w, 0), g \cdot p(w, \infty))$. One now verifies that all the desired properties of B, $v : B \times G \to G \times G$ and Σ hold.

Let $U \subset G$ be the non-empty open subset given by lemma 4, for the morphism $f : X \to H$ and the subvariety $Y \subset H$ (thus, for all $g \in U$, the subscheme $f^{-1}(gY)$ is purely of codimension 2 in $X - X_{\text{sing}}$, etc.). Consider the subset $W \subset v^{-1}(U \times U) \subset B \times G$ of points (w, g) such that

(i) $\widetilde{\Sigma}_{(w,g)} = (f \times 1_{\mathbb{P}^1})^{-1}(\Sigma_{(w,g)}) \subset X \times \mathbb{P}^1$ is purely of codimension 2 in $X \times \mathbb{P}^1$;
(ii) $\widetilde{\Sigma}_{(w,g)} \cap X_{\text{nc}} \times \mathbb{P}^1 = \emptyset$, where X_{nc} is the (finite) subset of non-Cohen–Macaulay points of X (since X is a reduced surface, it is Cohen–Macaulay in codimension 1, and the set of Cohen–Macaulay points of X is open);
(iii) $(\Sigma_{(w,g)})_{\text{sing}} \cap (f(X_{\text{sing}}) \times \mathbb{P}^1) = \emptyset$.

This is clearly a constructible subset of $B \times G$ (in (iii), note that $(\Sigma_{(w,g)})_{\text{sing}}$ is the fibre over (w, g) of the non-smooth locus of $\Sigma \to B \times G$). Note that for $(w, g) \in W$, any point of $(X_{\text{sing}} \times \mathbb{P}^1) \cap \widetilde{\Sigma}_{(w,g)}$ is a Cohen–Macaulay point of $X \times \mathbb{P}^1$, and lies over a smooth point of $\Sigma_{(w,g)}$. Since the ideal of $\Sigma_{(w,g)}$ in $H \times \mathbb{P}^1$ at such an image point is a complete intersection of height 2, we see that the ideal of $\widetilde{\Sigma}_{(w,g)}$ in $X \times \mathbb{P}^1$ is of height 2, generated by 2 elements, at each point of $(X_{\text{sing}} \times \mathbb{P}^1) \cap \widetilde{\Sigma}_{(w,g)}$. We deduce that $\widetilde{\Sigma}_{(w,g)}$ is a local complete intersection in $X \times \mathbb{P}^1$ at these points. Since the fibre of $\Sigma_{(w,g)} \to \mathbb{P}^1$ over 0 (or ∞) is a translate $g'Y$ by an element $g' \in U$, the scheme $f^{-1}(g'Y)$ is a finite subscheme of $X - X_{\text{sing}}$, and $\mathcal{O}_X$ and $\mathcal{O}_{g'Y}$ are Tor-independent over $\mathcal{O}_H$; since $f^{-1}(g'Y)$ is also the fibre of $\widetilde{\Sigma}_{(w,g)} \to \mathbb{P}^1$ over 0 (or ∞, respectively), lemma 5 implies that $\widetilde{\Sigma}_{(w,g)} \to \mathbb{P}^1$ is flat over 0 (and over ∞).

For any $(w, g) \in W$, we thus have that $\widetilde{\Sigma}_{(w,g)}$ determines an element of $R^2(X, X_{\text{sing}})$, namely $[(p_1)_*(\widetilde{\Sigma}_{(w,g)} \otimes k(0))] - [(p_1)_*(\widetilde{\Sigma}_{(w,g)} \otimes k(\infty))]$, which is just $[f^{-1}(g_1 Y)] - [f^{-1}(g_2 Y)]$, where $v(w, g) = (g_1, g_2)$. Hence for any $(g_1, g_2) \in v(W) \subset U \times U \subset G \times G$, we have that $f^{-1}(g_1 Y)$ and $f^{-1}(g_2 Y)$ determine the same element of $CH^2(X)$.

We next claim that W contains an open (dense) subset of $B \times G$. Fix $(w, g) \in B \times G$; for $g' \in G$, let $Z(g')$ denote the pullback $(f \times 1_{\mathbb{P}^1})^{-1}(g'\Sigma_{(w,g)})$. Then, by lemma 6, there exists a non-empty open subset $U_2 \subset G$ such that for $g' \in U_2$, we have:

(i) $Z(g')$ is purely of codimension 2 in $X \times \mathbb{P}^1$, and intersects $X_{\text{sing}} \times \mathbb{P}^1$ and $X \times \{0, \infty\}$ properly, and is disjoint from $X_{\text{sing}} \times \{0, \infty\}$, and $X_{\text{nc}} \times \mathbb{P}^1$;
(ii) $Z(g')$ is flat over a neighbourhood of $\{0, \infty\}$ in $\mathbb{P}^1$;
(iii) $Z(g') \otimes k(0) = f^{-1}(g'g \cdot p(w, 0)Y)$ and $Z(g') \otimes k(\infty) = f^{-1}(g'g \cdot p(w, \infty)Y)$;
(iv) the ideal of $Z(g')$ in $X \times \mathbb{P}^1$ is generated by a regular sequence at all points of $Z(g') \cap X_{\text{sing}} \times \mathbb{P}^1$.

We may further suppose, by shrinking U_2 if necessary, that $g'g \cdot p(w, 0)$ and $g'g \cdot p(w, \infty)$ are in U for all $g' \in U_2$. As $g'\Sigma_{(w,g)} = \Sigma_{(w,g'g)}$, we deduce that for each $w \in B$, there exists an open dense subset $U(w) \subset G$ such that $\{w\} \times U(w) \subset W$. Since W is constructible, we deduce that W contains a dense open subset of $B \times G$.

Thus $v(W)$ is a dense, constructible subset of $G \times G$. Hence there exists $g \in U$ and an open dense subset $V \subset G$ such that $\{g\} \times V \subset v(W)$. Then for any $g_1, g_2 \in V$, the cycles $[f^{-1}(g_1 Y)]$, $[f^{-1}(g_2 Y)]$ are rationally equivalent in $CH^2(X)$, since both are rationally equivalent to $[f^{-1}(gY)]$. □

Lemma 8. *(a) Let H be a quasi-projective homogenous space for a linear algebraic group G, and let $f : X \to H$ be a morphism. Then there is a well-defined homomorphism on*

Chow rings $f^* : CH^*(H) \to CH^*(X)$, such that the following properties hold.

(i) *If Y is a Cartier divisor on H, then $f^*[Y] = [Z]$ for any Cartier divisor Z on X with $\mathcal{O}_X(Z) \cong f^*\mathcal{O}_H(Y)$.*
(ii) *If Y is an irreducible subvariety of H of codimension 2, then $f^*[Y] = [f^{-1}(gY)]$, where g is any element of the open set V given by lemma 7.*

(b) Further, let H' be a quasi-projective homogeneous space for another linear algebraic group G', such that there is a homomorphism of algebraic groups $\rho : G \to G'$, and there is a morphism $h : H \to H'$ which is G-equivariant.

Then there is a commutative triangle

$$\begin{tikzcd} CH^*(H') \arrow[rr, "(f\circ h)^*"] \arrow[dr, "h^*"'] & & CH^*(X) \\ & CH^*(H) \arrow[ur, "f^*"'] & \end{tikzcd}$$

Proof. (a) Suppose $f : X \to H$ is a morphism. Then we have obvious homomorphisms $f^* : CH^0(H) \to CH^0(X)$ and $CH^1(H) \to CH^1(X)$. If Y is a reduced irreducible subvariety of H of codimension 2, then from lemma 7, there is a dense open subset $V \subset G$ such that (i) for each $g \in V$, the cycle $f^{-1}(gY)$ is a 0-cycle supported in $X - X_{\text{sing}}$, and (ii) for any $g_1, g_2 \in V$, the 0-cycles $[f^{-1}(g_1Y)]$ and $[f^{-1}(g_2Y)]$ are rationally equivalent. Hence $[Y] \mapsto [f^{-1}(gY)]$, for any $g \in V$, determines a well-defined homomorphism $f^* : Z^2(H) \to CH^2(X)$, where $Z^2(H)$ is the group of codimension 2 cycles on H.

To prove that f^* induces a map on $CH^2(H)$, we must show that if $Z \subset H \times \mathbb{P}^1$ is a reduced, codimension 2 subscheme flat over $\mathbb{P}^1$, and if $Y_1 = Z \cdot H \times \{0\}$, $Y_2 = Z \cdot H \times \{\infty\}$, then $f^*([Y_1]) = f^*([Y_2])$. From lemma 6, we see that there is an open dense subset $W \subset G$ such that for $g \in W$, the pullback $f^{-1}(gZ) \subset X \times \mathbb{P}^1$ defines a rational equivalence between $(p_1)_*(f^{-1}(gZ) \cdot X \times \{0\})$ and $(p_1)_*(f^{-1}(gZ) \cdot X \times \{\infty\})$ (*i.e.*, these are well-defined 0-cycles supported on $X - X_{\text{sing}}$, whose difference lies in $R^2(X, X_{\text{sing}}) = R^2(X)$). If V_1, V_2 are the open subsets corresponding, by lemma 7, to the cycles Y_1, Y_2 respectively, then we may take $g \in W \cap V_1 \cap V_2$; then we see that

$$[(p_1)_*(f^{-1}(gZ) \cdot X \times \{0\})] = f^*([Y_1]), \ [(p_1)_*(f^{-1}(gZ) \cdot X \times \{\infty\})] = f^*([Y_2]).$$

Hence $f^*([Y_1]) = f^*([Y_2])$ in $CH^2(X)$, as desired.

Next, to prove that $f^* : CH^*(H) \to CH^*(X)$ is a ring homomorphism, it suffices to show that if D_1, D_2 are two sufficiently ample, effective, smooth Cartier divisors on H which intersect transversely in a smooth subvariety Y of pure codimension 2 in H, then $f^*([D_1]) \cdot f^*([D_2]) = f^*([Y])$ in $CH^*(X)$. This is because such divisor classes $[D_i]$ generate Pic (H).

For all g in an open dense subset of G, the pullbacks $f^{-1}(gD_1)$, $f^{-1}(gD_2)$ may be assumed to be effective Cartier divisors on X which have finite intersection with X_{sing}, by lemma 4, while by lemma 7, we may also assume that $f^{-1}(gY)$ is a 0-cycle supported on $X - X_{\text{sing}}$, such that $f^*([Y]) = [f^{-1}(gY)]$. Now by the definition of the product $CH^1(X)^{\otimes 2} \to CH^2(X)$, we see that $f^*([D_1]) \cdot f^*([D_2]) = f^*([Y])$ holds in $CH^*(X)$. This completes the proof of (a) of the lemma.

To prove (b), it suffices to prove that $f^* \circ h^* = (h \circ f)^*$ on $CH^2(H')$. Further, it suffices to prove $f^* \circ h^*([Y]) = (h \circ f)^*([Y])$ for irreducible codimension 2 subvarieties $Y \subset H'$ such that

(i) $h^{-1}(Y)$ is a subscheme of H of pure codimension 2, and $h^*([Y]) = [h^{-1}(Y)]$ in $CH^2(H)$;
(ii) $[(f \circ h)^{-1}(Y)]$ is a 0-cycle on X, supported on $X - X_{\text{sing}}$, and $(f \circ h)^*([Y]) = [(f \circ h)^{-1}(Y)]$ in $CH^2(X)$;
(iii) there is an open dense subset $V' \subset G'$, containing the identity element of G', such that the properties (i), (ii) hold also for the cycle $g'Y$, for all $g' \in V'$.

This follows by applying lemmas 4 and 7 to the morphisms f and $h \circ f$. By lemma 7 and corollary 1 applied to $f : X \to H$, there is an open dense subset $V \subset G$ such that for all $g \in V$, the subscheme $f^{-1}(h^{-1}(Y)) \subset X - X_{\text{sing}}$ is 0-dimensional, such that $f^*[h^{-1}(Y)] = [(f^{-1}(g \cdot h^{-1}(Y))$ for all $g \in V$. Now $\rho^{-1}(V')$ is a non-empty open subset of G; choosing $g \in V \cap \rho^{-1}(V')$, we have a string of equalities in $CH^2(X)$,

$$
\begin{aligned}
f^* \circ h^*([Y]) &= f^*([h^{-1}(Y)]) = [f^{-1}(g \cdot h^{-1}(Y))] = [f^{-1}(h^{-1}(\rho(g)Y))] \\
&= [(f \circ h)^{-1}(\rho(g)Y)] = (f \circ h)^*([Y]),
\end{aligned}
$$

where the last equality is because $\rho(g) \in V'$. □

Lemma 9. *Let H be a quasi-projective homogeneous space for a linear algebraic group G, and $f_0, f_\infty : X \to H$ morphisms. Suppose there is a morphism $F : X \times \mathbb{P}^1 \to H$ such that $F(x, 0) = f_0(x)$, $F(x, \infty) = f_\infty(x)$. Then*

$$f_0^* = f_\infty^* : CH^*(H) \to CH^*(X).$$

Proof. It suffices, as usual, to show that $f_0^* = f_\infty^*$ on $CH^2(H)$. Let $Y \subset H$ be an irreducible subvariety of codimension 2.

By lemmas 4 and 7 applied to each of the maps f_0, f_∞, there is an open dense subset $U \subset G$ such that for all $g \in U$, the subschemes $f_0^{-1}(gY), f_\infty^{-1}(gY)$ yield 0-cycles supported in $X - X_{\text{sing}}$, which represent the rational equivalence classes $f_0^*[Y]$ and $f_\infty^*[Y]$ respectively.

Let $\widetilde{F} : X \times \mathbb{P}^1 \to H \times \mathbb{P}^1$ be given by $\widetilde{F}(x, t) = (F(x, t), t)$. By lemma 4 applied to $\widetilde{F}$, there is an open subset $\widetilde{U} \subset G \times \operatorname{Aut} \mathbb{P}^1$ such that for $(g, \sigma) \in G \times \operatorname{Aut}(\mathbb{P}^1)$, the subscheme $\widetilde{F}^{-1}((g, \sigma) \cdot (Y \times \mathbb{P}^1))$ is of pure codimension 2 in $X \times \mathbb{P}^1$, and intersects $X_{\text{sing}} \times \mathbb{P}^1$ in a finite set of points. Further, the ideal of $\widetilde{F}^{-1}((g, \sigma) \cdot (Y \times \mathbb{P}^1))$ in $X \times \mathbb{P}^1$ is a complete intersection at all points of intersection with $X_{\text{sing}} \times \mathbb{P}^1$.

Now $(g, \sigma) \cdot (Y \times \mathbb{P}^1) = gY \times \mathbb{P}^1$. The image of $\widetilde{U}$ under the projection $G \times \operatorname{Aut}(\mathbb{P}^1) \to G$ is a dense open subset in G. Hence there exists $(g, \sigma) \in \widetilde{U}$ with $g \in U$, so that $\widetilde{F}^{-1}((g, \sigma) \cdot (Y \times \mathbb{P}^1)) \to \mathbb{P}^1$ has fibres over $0, \infty$ which are 0-cycles supported in $X - X_{\text{sing}}$, and such that $\mathcal{O}_X$ is Tor-independent from $\mathcal{O}_{gY}$, with respect to the $\mathcal{O}_H$-module structures on $\mathcal{O}_X$ induced by f_0 as well as f_∞. Hence $\widetilde{F}^{-1}((g, \sigma) \cdot (Y \times \mathbb{P}^1)) \to \mathbb{P}^1$ is flat over a neighbourhood of $\{0, \infty\}$, by lemma 5. Thus $\widetilde{F}^{-1}((g, \sigma) \cdot (Y \times \mathbb{P}^1)) \subset X \times \mathbb{P}^1$ determines an element of $R^2(X, X_{\text{sing}})$, which we compute to be the 0-cycle $[f_0^{-1}(gY)] - [f_\infty^{-1}(gY)]$. Since $g \in U$, this means that $[f_0^{-1}(gY)] = f_0^*[Y]$, and $[f_\infty^{-1}(gY)] = f_\infty^*[Y]$ in $CH^2(X)$. Hence these two elements of $CH^2(X)$ are equal. □

5. Chern classes

In this section, we construct Chern classes for locally free sheaves on X, and prove that they descend to give a homomorphism $c : K_0(X) \to CH^*(X)^*$ to the multiplicative group of units in the Chow ring. We will assume further that X is *connected*, since the general

case can be deduced immediately from this one. Then any locally free sheaf on X has a well-defined rank, which is a non-negative integer.

For any locally free sheaf $\mathcal{E}$, we define $c_0(\mathcal{E}) = 1$; if $\mathcal{E}$ has rank r, then $\overset{r}{\wedge}\mathcal{E}$ is an invertible sheaf, and hence determines a well-defined class of Cartier divisors D on X such that $\mathcal{O}_X(D) \cong \overset{r}{\wedge}\mathcal{E}$; we define $c_1(\mathcal{E}) = [D]$. Note that if

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is an exact sequence of locally free sheaves, of ranks r', r, r'' respectively, then we have an isomorphism of invertible sheaves

$$\overset{r'}{\wedge}\mathcal{E}' \otimes \overset{r''}{\wedge}\mathcal{E}'' \cong \overset{r}{\wedge}\mathcal{E},$$

and so $c_1(\mathcal{E}) = c_1(\mathcal{E}') + c_1(\mathcal{E}'')$. Also, if $f : X \to H$ is a morphism to a homogeneous space H, and $\mathcal{M}$ is an invertible sheaf on H, then $f^*(c_1(\mathcal{M})) = c_1(f^*\mathcal{M})$.

Let $\mathbb{G}(r,N)$ denote the Grassmannian of r-dimensional quotients of the N-dimensional vector space k^N. Let $\mathcal{Q}_{r,N}$ be the universal quotient bundle on $\mathbb{G}(r,N)$, with its tautological sections $\mathrm{s}_1, \ldots, \mathrm{s}_N$ which generate $\mathcal{Q}_{r,N}$. Then we have the following *universal property* of the Grassmannian: if Z is a k-scheme, $\mathcal{E}$ a locally free sheaf of rank r, and $s_1, \ldots, s_N$ a set of global sections which generate $\mathcal{E}$, then we have a unique "classifying morphism" $f : Z \to \mathbb{G}(r,N)$ and a unique isomorphism $\psi : f^*\mathcal{Q}_{r,N} \to \mathcal{E}$ such that $\psi(f^*\mathrm{s}_i) = s_i$.

Lemma 10. *Let $\mathcal{E}$ be a locally free sheaf on X, and $\{s_1, \ldots, s_M\}$, $\{t_1, \ldots, t_N\}$ two sets of global sections of $\mathcal{E}$, each of which generates $\mathcal{E}$. Let $f_1 : X \to \mathbb{G}(r,M)$, $f_2 : X \to \mathbb{G}(r,N)$ be the corresponding classifying morphisms. Then*

$$f_1^*(c_2(\mathcal{Q}_{r,M})) = f_2^*(c_2(\mathcal{Q}_{r,N}))$$

in $CH^2(X)$.

Proof. On $X \times \mathbb{P}^1$, the locally free sheaf

$$\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) = p_1^*\mathcal{E} \otimes_{\mathcal{O}_{X\times\mathbb{P}^1}} p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$$

has global sections $s_1 \boxtimes u, \ldots, s_M \boxtimes u, t_1 \boxtimes v, \ldots, t_N \boxtimes v$ where u, v is the standard basis for global sections of $\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{Q}_{1,2}$. It is easy to see that $\mathcal{E} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ is generated by these global sections (for example, on $X \times (\mathbb{P}^1 - \{u = 0\})$, $s_1 \boxtimes u, \ldots, s_M \boxtimes u$ generate, since $s_1, \ldots, s_M$ generate $\mathcal{E}$). Thus we have a corresponding classifying morphism $F : X \times \mathbb{P}^1 \to \mathbb{G}(r, M+N)$. By construction, if $0, \infty$ in $\mathbb{P}^1$ are given by $u \neq 0, v = 0$ and $v \neq 0, u = 0$, then we have commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{f_1} & \mathbb{G}(r,M) \\ i_0 \downarrow & & \downarrow \alpha \\ X \times \mathbb{P}^1 & \xrightarrow{F} & \mathbb{G}(r,M+N) \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{f_2} & \mathbb{G}(r,N) \\ i_\infty \downarrow & & \downarrow \beta \\ X \times \mathbb{P}^1 & \xrightarrow{F} & \mathbb{G}(r,M+N) \end{array}$$

where $i_t : X \to X \times \mathbb{P}^1$ is the inclusion $x \mapsto (x,t)$, for any $t \in \mathbb{P}^1$. The map $\alpha : \mathbb{G}(r,M) \to \mathbb{G}(r,M+N)$ is the classifying morphism for the locally free sheaf $\mathcal{Q}_{r,M}$ and the $M+N$ sections $\mathrm{s}_1, \ldots, \mathrm{s}_M, 0, \ldots, 0$ (with N sections which are identically 0); it is an embedding. The map β is similarly defined. In particular, $\alpha^*(\mathcal{Q}_{r,M+N}) = \mathcal{Q}_{r,M}$, and $\beta^*\mathcal{Q}_{r,M+N} = \mathcal{Q}_{r,N}$. Further, there are obvious (faithful) representations $\mathrm{GL}_M(k) \to \mathrm{GL}_{M+N}(k)$, $GL_N(k) \to$

$\mathrm{GL}_{M+N}(k)$ such that α, β become equivariant morphisms for $\mathrm{GL}_M(k)$, $\mathrm{GL}_N(k)$ respectively.

From the theory of Chow rings and Chern classes on non-singular varieties, we have that $\alpha^*(c_2(\mathcal{Q}_{r,M+N})) = c_2(\mathcal{Q}_{r,M})$ and $\beta^*(c_2(\mathcal{Q}_{r,M+N})) = c_2(\mathcal{Q}_{r,N})$. Now applying lemmas 8 and 9, we have equalities

$$f_1^*(c_2(\mathcal{Q}_{r,M})) = f_1^*\alpha^*((c_2(\mathcal{Q}_{r,M+N})) = f_2^*\beta^*(c_2(\mathcal{Q}_{r,M+N})) = f_2^*(c_2(\mathcal{Q}_{r,N})). \quad \square$$

Lemma 11. *Let $\mathcal{E}$ be a locally free sheaf of rank r on X generated by sections $s_1, \ldots, s_M$, and let $\mathcal{L}$ be an invertible sheaf generated by sections $t_1, \ldots, t_N$. Let $f : X \to \mathbb{G}(r, M)$ be the classifying morphism for $\mathcal{E}$ and the sections s_i, and let $h : X \to \mathbb{G}(r, MN)$ be the classifying morphism for $\mathcal{E} \otimes \mathcal{L}$ and the sections $s_i \otimes t_j$. Then*

$$h^*(c_2(\mathcal{Q}_{r,MN})) = f^*(c_2(\mathcal{Q}_{r,M})) + (r-1)c_1(\mathcal{E}) \cdot c_1(\mathcal{L}) + \binom{r}{2} c_1(\mathcal{L})^2.$$

Proof. Let $g : X \to \mathbb{P}^{N-1}$ be the morphism determined by $\mathcal{L}$ and the sections t_j; then we have a morphism $(f, g) : X \to \mathbb{G}(r, M) \times \mathbb{P}^{N-1}$. We have a classifying morphism $F : \mathbb{G}(r, M) \times \mathbb{P}^{N-1} \to \mathbb{G}(r, MN)$ corresponding to the locally free sheaf $\mathcal{Q}_{r,M} \boxtimes \mathcal{O}_{\mathbb{P}^{N-1}}(1)$, and the sections $\mathbf{s}_i \boxtimes \mathbf{t}_j$, $1 \leq i \leq M$, $1 \leq j \leq N$ (where $\mathbf{s}_i$ and $\mathbf{t}_j$ are the tautological sections); clearly $h = F \circ (f, g) : X \to \mathbb{G}(r, MN)$. Notice that $F : \mathbb{G}(r, M) \times \mathbb{P}^{N-1} \to \mathbb{G}(r, MN)$ is $\mathrm{GL}_M(k) \times \mathrm{GL}_N(k)$-equivariant, for the tensor product representation $\rho : \mathrm{GL}_M(k) \times \mathrm{GL}_N(k) \to \mathrm{GL}_{MN}(k)$.

If $p_1 : \mathbb{G}(r, M) \times \mathbb{P}^{N-1} \to \mathbb{G}(r, M)$, $p_2 : \mathbb{G}(r, M) \times \mathbb{P}^{N-1} \to \mathbb{P}^{N-1}$ are the projections, then the p_i are $\mathrm{GL}_M(k) \times \mathrm{GL}_N(k)$-equivariant, for the actions on $\mathbb{G}(r, M)$ and $\mathbb{P}^{N-1}$ induced by the projections $\mathrm{GL}_M(k) \times \mathrm{GL}_N(k)$ onto the two factors.

From the formalism of Chern classes of locally free sheaves on non-singular varieties, we have an equation in $CH^*(\mathbb{G}(r, M) \times \mathbb{P}^{N-1})$

$$\begin{aligned} F^*(c_2(\mathcal{Q}_{r,MN})) &= c_2(\mathcal{Q}_{r,M} \boxtimes \mathcal{O}_{\mathbb{P}^{N-1}}(1)) \\ &= p_1^*(c_2(\mathcal{Q}_{r,M})) + (r-1)p_1^*(c_1(\mathcal{Q}_{r,M})) \cdot p_2^*(c_1(\mathcal{O}_{\mathbb{P}^{N-1}}(1))) \\ &\quad + \binom{r}{2} p_2^*(c_1(\mathcal{O}_{\mathbb{P}^{N-1}}(1)))^2. \end{aligned}$$

We have a commutative diagram

$$\begin{tikzcd}
 & \mathbb{G}(r, M) & \\
X \arrow[ur, "f"] \arrow[r, "(f,g)"] \arrow[dr, "g"'] & \mathbb{G}(r, M) \times \mathbb{P}^{N-1} \arrow[u, "p_1"'] \arrow[d, "p_2"] \arrow[r, "F"] & \mathbb{G}(r, MN) \\
 & \mathbb{P}^{N-1} &
\end{tikzcd}$$

from which, by lemma 8, we obtain the desired equation in $CH^*(X)$. $\square$

We can now define $c_2(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ of rank r on X, as follows. Choose an invertible sheaf $\mathcal{L}$ generated by its global sections, such that $\mathcal{E} \otimes \mathcal{L}$ is generated by its global sections, and let $s_1, \ldots, s_N$ be global sections of $\mathcal{E} \otimes \mathcal{L}$ which generate it. For example, we may choose $\mathcal{L}$ to be a high power of an ample invertible sheaf. Let $f : X \to \mathbb{G}(r, N)$ be the classifying morphism for $\mathcal{E} \otimes \mathcal{L}$ and the sections s_i. Then

define

$$c_2(\mathcal{E}) = f^*(c_2(\mathcal{Q}_{r,M})) - (r-1)c_1(\mathcal{E}) \cdot c_1(\mathcal{L}) - \binom{r}{2} c_1(\mathcal{L})^2.$$

From lemmas 10 and 11, we claim that this definition is *independent of the choices made.* To verify independence of $\mathcal{L}$, let $\mathcal{M}$ be another such choice. Then we apply lemma 11 to $(\mathcal{E} \otimes \mathcal{L}) \otimes \mathcal{M}$ and to $(\mathcal{E} \otimes \mathcal{M}) \otimes \mathcal{L}$.

We can now also define the *total Chern class*

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}).$$

Lemma 12. *(a) Let $h : X \to H$ be a morphism to a projective homogeneous space for a connected linear algebraic group G, and let $\mathcal{Q}$ be a G-homogeneous locally free sheaf on H. Then $c(h^*\mathcal{Q}) = h^*(c(\mathcal{Q}))$ in $CH^*(X)$.*
(b) Furthermore, every locally free sheaf $\mathcal{E}$ on X arises as $h^*\mathcal{Q}$ for some $h : X \to H$ and some $\mathcal{Q}$.

Proof. (a) It suffices to prove the functoriality property for c_2, since it is obvious for the other Chern classes. Since H is quasi-projective, we can find a very ample G-homogeneous invertible sheaf $\mathcal{L}$ generated by global sections such that $\mathcal{Q} \otimes \mathcal{L}$ is also generated by global sections; let $s_1, \ldots, s_N$ be a k-basis for the (finite dimensional) space of global sections $\Gamma(H, \mathcal{Q} \otimes \mathcal{L})$. Let $g : H \to \mathbb{G}(r,N)$ be the classifying morphism for $\mathcal{Q} \otimes \mathcal{L}$ and the sections s_i. Since $\mathcal{Q} \otimes \mathcal{L}$ is also G-homogeneous on H, there is a natural associated representation of G on the global sections $\rho : G \to \mathrm{GL}_N(k)$ such that the classifying morphism $g : H \to \mathbb{G}(r,N)$ is G-equivariant (where G acts on $\mathbb{G}(r,N)$ through the representation ρ).

Now $h^*\mathcal{L}$ is an invertible sheaf on X generated by global sections, and $h^*\mathcal{Q} \otimes h^*\mathcal{L}$ is generated by the pull-backs $h^*(s_i)$ of the sections of $\mathcal{Q} \otimes \mathcal{L}$. The classifying morphism for $h^*\mathcal{Q} \otimes h^*\mathcal{L}$ and the sections $h^*(s_i)$ is clearly just $f = g \circ h : X \to \mathbb{G}(r,N)$. Now (a) of the lemma follows from lemmas 8, 10 and 11.

To prove (b), suppose $\mathcal{E}$ has rank r. Choose $\mathcal{L}$ generated by global sections $t_1, \ldots, t_M$ such that $\mathcal{E} \otimes \mathcal{L}$ is generated by global sections $s_1, \ldots, s_N$. Let $f : X \to \mathbb{G}(r,N)$ be the classifying morphism for $\mathcal{E} \otimes \mathcal{L}$ and the sections s_i, and let $g : X \to \mathbb{P}^{M-1}$ be the classifying morphism for $\mathcal{L}$ and the sections t_j. Then $H = \mathbb{G}(r,N) \times \mathbb{P}^{M-1}$ is a projective G-homogeneous space, for $G = \mathrm{GL}_N(k) \times \mathrm{GL}_M(k)$, and $\mathcal{Q} = \mathcal{Q}_{r,N} \boxtimes \mathcal{O}_{\mathbb{P}^{M-1}}(-1)$ is a G-homogeneous locally free sheaf on H, such that for the morphism $h = (f,g) : X \to H$, we have $h^*\mathcal{Q} \cong \mathcal{E}$. □

Lemma 13. *Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be an exact sequence of locally free sheaves. Then $c(\mathcal{E}) = c(\mathcal{E}') \cdot c(\mathcal{E}'')$ in $CH^*(X)$.*

Proof. Let $\mathcal{L}$ be an invertible sheaf generated by its global sections such that $\mathcal{E} \otimes \mathcal{L}$ is generated by global sections $s_1, \ldots, s_N$. Then $\mathcal{E}'' \otimes \mathcal{L}$ is generated by the images of the s_i. Let $\mathbb{F}(r, r''; N)$ denote the flag variety of quotients $k^N \twoheadrightarrow V \twoheadrightarrow W$ with $\dim V = r$, $\dim W = r''$; then there is a classifying morphism $f : X \to \mathbb{F}(r, r''; N)$. Here $\mathbb{F}(r, r''; N)$ is a closed subscheme of $\mathbb{G}(r,N) \times \mathbb{G}(r'',N)$, and is a homogeneous space for $\mathrm{GL}_N(k)$ such that the projections $p : \mathbb{F}(r, r''; N) \to \mathbb{G}(r,N)$, $q : \mathbb{F}(r, r'', N) \to \mathbb{G}(r'',N)$ are $\mathrm{GL}_N(k)$-equivariant. There is an exact sequence of $\mathrm{GL}_N(k)$-homogeneous locally free sheaves on

$\mathbb{F}(r, r''; N)$,

$$0 \to \mathcal{S} \to p^*\mathcal{Q}_{r,N} \to q^*\mathcal{Q}_{r'',N} \to 0,$$

which pulls back under f to the exact sequence

$$0 \to \mathcal{E}' \otimes \mathcal{L} \to \mathcal{E} \otimes \mathcal{L} \to \mathcal{E}'' \otimes \mathcal{L} \to 0.$$

If $g : X \to \mathbb{P}^M$ is a classifying morphism for $\mathcal{L}$ and a suitable space of sections, then we have an exact sequence of locally free sheaves on $H = \mathbb{F}(r, r''; N) \times \mathbb{P}^M$,

$$0 \to \mathcal{S} \boxtimes \mathcal{O}_{\mathbb{P}^M}(-1) \to p^*\mathcal{Q}_{r,N} \boxtimes \mathcal{O}_{\mathbb{P}^M}(-1) \to q^*\mathcal{Q}_{r'',N} \boxtimes \mathcal{O}_{\mathbb{P}^M}(-1) \to 0.$$

Now H is a homogeneous space for $G = \mathrm{GL}_N(k) \times \mathrm{GL}_{M+1}(k)$, and under the morphism $(f, g) : X \to H$, the above exact sequence of G-homogeneous locally free sheaves pulls back to the given exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$. Since

$$c(p^*\mathcal{Q}_{r,N} \boxtimes \mathcal{O}_{\mathbb{P}^M}(-1)) = c(\mathcal{S} \boxtimes \mathcal{O}_{\mathbb{P}^M}(-1)) \cdot c(q^*\mathcal{Q}_{r'',N} \boxtimes \mathcal{O}_{\mathbb{P}^M}(-1))$$

in $CH^*(H)$, from the usual formalism of Chern classes on non-singular varieties, lemma 12 completes the proof of the lemma. □

Lemma 14. *Let $\pi : Z \to X$ be a projective morphism from a non-singular surface, such that $\pi^{-1}(X - X_{\text{sing}}) \to X - X_{\text{sing}}$ is quasi-finite. Then there is a ring homomorphism $\pi^* : CH^*(X) \to CH^*(Z)$, such that for any locally free sheaf $\mathcal{E}$ on X, we have $\pi^*(c(\mathcal{E})) = c(\pi^*(\mathcal{E}))$ (where the right side is defined in the usual way, for the locally free sheaf $\pi^*(\mathcal{E})$ on the non-singular variety Z).*

Proof. The homomorphisms $CH^0(X) \to CH^0(Z)$, $CH^1(X) \to CH^1(Z)$ are the obvious ones. We need only define a homomorphism on $CH^2(X)$. If $P \in X - X_{\text{sing}}$, define $\pi^*([P]) = [\pi^{-1}(P)]$, the class in $CH^2(Z)$ associated to the 0-cycle determined by $\pi^{-1}(P)$. Here, if $\pi^{-1}(P) = \emptyset$, then we set $[\pi^{-1}(P)] = 0$. This determines a well-defined homomorphism $\pi^* : Z^2(X) \to CH^2(Z)$. If $C \subset X$ is a Cartier curve, $h \in R(C)$, then either

(i) $\pi^{-1}(C - X_{\text{sing}}) = \emptyset$, in which case $\pi^*((h)_C) = 0$, or
(ii) $\tilde{C} = \overline{\pi^{-1}(C - X_{\text{sing}})} \subset Z$ is a 1-dimensional closed subscheme, and $\pi^*(h)$ is a rational function on $\tilde{C}$, such that $(\pi^*(h))_{\tilde{C}} = \pi^*((h)_C)$ as 0-cycles.

In either case, $\pi^*((h)_C) = 0$. Hence $\pi^* : CH^2(X) \to CH^2(Z)$ is well-defined. If D_1, D_2 are effective Cartier divisors on X, each with finite intersections with X_{sing}, such that $D_1 \cap D_2 \cap X_{\text{sing}} = \emptyset$, then $\pi^*(D_1)$, $\pi^*(D_2)$ are effective Cartier divisors on Z with finite intersection, and $\pi^*(D_1) \cdot \pi^*(D_2) = \pi^*(D_1 \cdot D_2)$ as 0-cycles. Hence π^* is a ring homomorphism.

If $\mathcal{E}$ is a locally free sheaf on X, choose (by lemma 12(b)) a morphism $f : X \to H$ to a projective G-homogeneous space, and a G-homogeneous locally free sheaf $\mathcal{Q}$ on H, such that $\mathcal{E} = f^*\mathcal{Q}$, and $c_i(\mathcal{E}) = f^*(c_i(\mathcal{Q}))$.

Now $(f \circ \pi)^*\mathcal{Q} = \pi^*\mathcal{E}$. We claim

$$(f \circ \pi)^* = \pi^* \circ f^* : CH^*(H) \to CH^*(Z).$$

This is clear on CH^0 and CH^1. Let Y be an irreducible subvariety of codimension 2 on H. Suppose $f^{-1}(Y)$ is a finite subscheme of $X - X_{\text{sing}}$, and $\mathcal{O}_X$, $\mathcal{O}_Y$ are Tor-independent over $\mathcal{O}_H$; then $(f \circ \pi)^{-1}(Y)$ is a 0-dimensional subscheme of Z, such that $\mathcal{O}_Z$ and $\mathcal{O}_Y$ are Tor-independent over $\mathcal{O}_H$ as well. This is because $Z - \pi^{-1}(X_{\text{sing}}) \to X - X_{\text{sing}}$ is a quasi-finite

morphism between non-singular surfaces, and is hence flat. Hence by corollary 1, after a suitable translation by an element $g \in G$, the class in $CH^2(Z)$ of the cycle $[(f \circ \pi)^{-1}(gY)]$ represents both $\pi^* f^*([Y])$ and $(f \circ \pi)^*([Y])$ (the latter is defined in terms of intersections multiplicites, which are alternating sums of lengths of Tor 's, which however vanish in positive degrees because of Tor-independence).

Now lemma 12, which is valid also for Z in place of X (as Z is non-singular) implies that

$$\pi^*(c_2(\mathcal{E})) = \pi^* f^*(c_2(\mathcal{Q})) = (f \circ \pi)^*(c_2(\mathcal{Q})) = c_2((f \circ \pi)^*(\mathcal{Q})) = c_2(\pi^*\mathcal{E}).$$

□

Lemma 15. *Let X be a surface, U an open subset, $j : U \to X$ the inclusion. Let $Z = X - U$ be the (reduced) complement, $i : Z \to X$ the inclusion. Then we have the following.*

(i) There is a natural homomorphism $j^* : CH^*(X) \to CH^*(U)$, such that for any locally free sheaf $\mathcal{E}$ on X, we have $c(j^*\mathcal{E}) = j^*(c(\mathcal{E}))$ in $CH^*(U)$.
(ii) If $X_{\text{sing}} \subset U$, then there is an exact sequence

$$CH_*(Z) \xrightarrow{i_*} CH^*(X) \xrightarrow{j^*} CH^*(U) \to 0,$$

where $CH_*(Z)$ denotes (graded) the Chow group of Fulton [Ful], and i_* is induced by the obvious map on cycles (in particular it does not preserve gradings).

Proof. The proof of (i) is easy, given our definitions, and is left to the reader.

In (ii), it is again easy to see that j^* is surjective, with kernel equal to the subgroup of classes of cycles supported on Z. Indeed, given any cycle class $\alpha = \sum_i n_i[Z_i]$ on X, if we set $\overline{\alpha} = \sum_i n_i[\overline{Z_i}]$, where $\overline{Z_i}$ is the closure of Z_i in X, then $j^*\overline{\alpha} = \alpha$. Similarly if a cycle δ on X restricts to a cycle on U which is rationally equivalent to 0 on U, then we may form the "closure" of the rational equivalence on X, which will yield a rational equivalence on X between δ and another cycle which will be supported on Z.

It remains to remark that if an r-dimensional cycle δ on Z is rationally equivalent to 0 on Z in the sense of Fulton, then either (a) $r = 1$ and the cycle is itself 0, or (b) $r = 0$, in which case $\delta = \sum_s (f_s)_{Z_s}$ is a sum of divisors of rational functions on 1-dimensional components Z_s of Z; then $C = \cup_s Z_s$ is a Cartier curve on X, and $f \in R(C)$ defined by $f|_{Z_s} = f_s$ has $(f)_C = \delta$, so that $[\delta] = 0$ in $CH^2(X)$ as well. □

6. The Riemann–Roch theorem

Any point $x \in X - X_{\text{sing}}$ has a well-defined class in $K_0(X)$, the Grothendieck group of locally free sheaves on X, which we identify with the Grothendieck group of coherent $\mathcal{O}_X$-modules with finite $\mathcal{O}_X$-homological dimension. Let $F_0K_0(X)$ be the subgroup generated by classes of points of $X - X_{\text{sing}}$. There is a natural surjective homomorphism $Z^2(X) \to F_0K_0(X)$. We first show that it induces a homomorphism $\psi : CH^2(X) \to F_0K_0(X)$.

Let $W \subset X \times \mathbb{P}^1$ be a subscheme defining an element of $R^2(X, X_{\text{sing}})$. Then $\mathcal{O}_W$ has finite homological dimension on $X \times \mathbb{P}^1$, since the ideal of W is a complete intersection at all singular points of $X \times \mathbb{P}^1$. We have an equality in $K_0(X \times \mathbb{P}^1)$

$$[\mathcal{O}_W \otimes \mathcal{O}_{X \times \{0\}}] = [\mathcal{O}_W] \cdot [\mathcal{O}_{X \times \{0\}}],$$

where the right side denotes the product in $K_0(X \times \mathbb{P}^1)$, since $W \to \mathbb{P}^1$ is flat over $\{0\}$ (in general, the product of classes of 2 sheaves of finite homological dimension would be

"the alternating sum of the classes of $\mathcal{T}or$ sheaves", *i.e.*, the class corresponding to the tensor product of 2 locally free resolutions). A similar formula holds for the class of $\mathcal{O}_W \otimes \mathcal{O}_{X\times\{\infty\}}$. But $[\mathcal{O}_{X\times\{0\}}] = [\mathcal{O}_{X\times\{\infty\}}]$ in $K_0(X \times \mathbb{P}^1)$, since $[0] = [\infty]$ in $K_0(\mathbb{P}^1)$, and if $q : X \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection, then q is flat, and $q^*[t] = [\mathcal{O}_{X\times\{t\}}]$ for any (closed) point $t \in \mathbb{P}^1$. Finally, since the projection $p : X \times \mathbb{P}^1 \to X$ is proper and flat, we have a push-forward map $p_* : K_0(X \times \mathbb{P}^1) \to K_0(X)$ which maps the class of a smooth point y to the class of its image $[p(y)] \in F_0K_0(X)$. In particular,

$$p_*([\mathcal{O}_W \otimes \mathcal{O}_{X\times\{0\}}]) = [p_*(W \cdot X \times \{0\})],$$
$$p_*([\mathcal{O}_W \otimes \mathcal{O}_{X\times\{\infty\}}]) = [p_*(W \cdot X \times \{\infty\})];$$

since the left sides in the above two equations are the same, so are the right sides. This shows that $[\delta] = 0$ for any $\delta \in R^2(X, X_{\rm sing})$, as desired.

By lemma 13, we deduce (as usual) that there are well-defined Chern class maps $c_i : K_0(X) \to CH^i(X)$, $0 \leq i \leq 2$, and a total chern class map $c : K_0(X) \to CH^*(X)$, such that

(i) $c_0(a) = 1$ for all $a \in K_0(X)$, so that $c(a)$ is a unit in $CH^*(X)$,
(ii) $c(a + b) = c(a)c(b)$ for all $a, b \in K_0(X)$, *i.e.*, c is a homomorphism from the additive group of $K_0(X)$ to the group of units of $CH^*(X)$.

PROPOSITION 2 (Riemann–Roch theorem)

The map $c_2 : F_0K_0(X) \to CH^2(X)$ is an isomorphism of abelian groups, with inverse $-\psi$.

Proof. We first note that any element of $F_0K_0(X)$ has vanishing c_1, since any invertible $\mathcal{O}_X$-module which is trivial on the complement of a finite subset of $X - X_{\rm sing}$ is in fact trivial. This implies easily that c_2 is a group homomorphism on $F_0K_0(X)$, since in general, for any $a, b \in K_0(X)$, we have a formula relating total Chern classes $c(a + b) = c(a)c(b)$. Hence it suffices to prove that $c_2 \circ \psi$ is multiplication by -1; it suffices to verify this on classes of smooth points. From lemma 15, it suffices to verify this when X is projective.

Let $x \in X - X_{\rm sing}$, and let X_0 be the irreducible component of X containing x. Let $\pi : Z \to X$ be a resolution of singularities of X_0, composed with the inclusion $X_0 \hookrightarrow X$. Then from lemma 14, we have

$$\pi^*(c_2 \circ \psi([x])) = \pi^*(c_2([\mathcal{O}_{\{x\}}])) = c_2(\pi^*([\mathcal{O}_{\{x\}}]))$$
$$= c_2([\mathcal{O}_{\{\pi^{-1}(x)\}}] = -[\pi^{-1}(x)] \in CH^2(Z).$$

Here the last equality is by the analogous Riemann–Roch theorem for a nonsingular surface. On the other hand, $\ker(CH^2(X) \to CH^2(X - \{x\})$ is clearly the (free) cyclic subgroup generated by $[x]$; since $\psi[x] \in \ker(K_0(X) \to K_0(X - \{x\})$, lemma 15 gives

$$c_2 \circ \psi[x] \in \ker(CH^2(X) \to CH^2(X - \{x\})).$$

Hence $c_2 \circ \psi[x] = m[x]$ for some integer m. Applying π^*, we deduce that $m = -1$. □

References

[BiS] Biswas J and Srinivas V, *Roitman's theorem for singular projective varieties, Compositio Math.* (to appear)
[Co] Collino A, Quillen's K-theory and algebraic cycles on almost non-singular varieties, *Ill. J. Math.* **25** (1981) 654–666

[Ful] Fulton W, Intersection Theory, *Ergeb. Math.* 3(2), Springer-Verlag (1984)

[H] Hartshorne R, *Algebraic Geometry*, Grad. Texts in Math. No. 52, Springer-Verlag (1977)

[L] Levine M, Bloch's formula for a singular surface, *Topology* **24** (1985) 165–174

[L1] Levine M, *A geometric theory of the Chow ring on a singular variety*, preprint

[LW] Levine M and Weibel C, Zero-cycles and complete intersections on singular varieties, *J. Reine Ang. Math.* 359 (1985) 106–120

[MS] Murthy M P and Swan R, Vector bundles over affine surfaces, *Invent. Math.* **36** (1976) 125–165

[PW] Pedrini C and Weibel C A, *K*-theory and Chow groups on singular varieties, *AMS Contemp. Math.* **55** (1986) 339–370

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 3, October 1998, pp. 251–271.

Absolute N_{q_α}-summability of the series conjugate to a Fourier series

A K SAHOO

Department of Mathematics, Government Kolasib College, Post Box No-20, Kolasib 796 081, Mizoram, India

MS received 1 May 1997; revised 16 December 1997 and 16 May 1998

Abstract. The object of the paper is to study the absolute Nq_α-summability of the series conjugate to a Fourier series, generalising a known result.

Keywords. Fourier series; conjugate series; kernel cesàro sum.

1. Introduction

In the year 1921 [6] F Nevanlinna suggested and discussed an interesting method of summation named as N_q-method. In 1932 [4] A F Moursund applied this method for the summation of Fourier series and allied series. In 1933 in a paper [5] he discussed N_{q_p}-method of summation, where p is a positive integer and applied it to p-th derived Fourier series. In his Ph D thesis [10] Samal studied absolute N_{q_p}-method of summation of p-th derived Fourier series. Samal in chapter-V of his Ph D thesis [10] extended the N_{q_p}-method of summation to N_{q_α}-method of summation for any $\alpha \geq 0$ and applied $|N_{q_\alpha}|$-method of summation to the Fourier series. In this paper we shall apply this method of summation to the series conjugate to a Fourier series.

DEFINITION 1 ([5], [10]).

Let $F(w)$ be a function of a continuous parameter w defined for all $w > 0$. The N_{q_α}-method of summation $(\alpha > 0)$ consists in forming the N_{q_α}-transform or mean of $F(w)$

$$N_{q_\alpha} F(w) \equiv \int_0^1 q_\alpha(t) F(wt) \mathrm{d}t.$$

If $\lim_{w \to \infty} N_{q_\alpha} F(w) = S$, then we say that $F(w)$ is summable by N_{q_α}-method to the sum S. In short we write

$$\lim_{w \to \infty} F(w) = S(N_{q_\alpha}),$$

where the class of functions $q_\alpha(t)$ is such that when $\alpha \geq 1$, all conditions (i)–(vii) hold and in case $0 \leq \alpha < 1$, all hold except conditions (iii) and (iv). In later case $(0 \leq \alpha < 1)\, q_\alpha(t) \geq 0$ and monotonic increasing (the case $h = 0$ of (vi))

(i) $\qquad q_\alpha(t) \geq 0$ for $0 \leq t \leq 1$

(ii) $\qquad \int_0^1 q_\alpha(t) \mathrm{d}t = 1$

(iii) $(\mathrm{d}/\mathrm{d}t)^{\beta} q_\alpha(t)$ exists and absolutely continuous for $0 \leq t \leq 1$, where $[\alpha] = h$ and $\beta = 0, 1, 2, \ldots, h-1$

(iv) $(\mathrm{d}/\mathrm{d}t)^{\beta} q_\alpha(t) = 0$ for $t = 1, \beta = 0, 1, 2, \ldots, h-1$

(v) $(\mathrm{d}/\mathrm{d}t)^{h} q_\alpha(t)$ exists for $0 < t < 1$

(vi) $q^h(t) \geq 0$ and monotonic increasing for $0 < t < 1$, where $q^h(t) = (-1)^h (\mathrm{d}/\mathrm{d}t)^h q_\alpha(t)$

(vii) $$\int_0^t \frac{Q_h(u)}{u^{1+\alpha-h}} \mathrm{d}u = 0\left(\frac{Q_h(t)}{t^{\alpha-h}}\right) \text{ for } 0 < t \leq \pi,$$

where

$$Q_h(t) = \int_{1-t}^{1} q^h(u)\mathrm{d}u. \tag{1}$$

Also we set

$$Q(t) = \int_{1-t}^{1} q_\alpha(u)\mathrm{d}u. \tag{2}$$

The following equivalent definition of N_{q_α}-method is due to Samal ([9], [10]).

DEFINITION 2 ([9], [10]).

Suppose that $q_\alpha(t)$ is the class of functions as defined in Definition 1, and $\sum_{n=0}^{\infty} u_n$ is an infinite series with $S(w) = \sum_{n \leq w} u_n$.

If

$$\lim_{w \to \infty} \sum_{n \leq w} u_n Q\left(1 - \frac{n}{w}\right) = l. \tag{3}$$

We say that $\sum u_n$ is summable by N_{q_α}-method to the sum l. In short we write

$$\sum u_n = l(N_{q_\alpha}).$$

Further the series $\sum u_n$ is said to be $|N_{q_\alpha}|$-summable if

$$\int_A^{\infty} \frac{\mathrm{d}w}{w^2} \left| \sum_{n \leq w} n u_n q_\alpha(n/w) \right| < \infty, \tag{4}$$

where A is a positive constant.

For $\alpha = 0$, the method reduces to original N_q-method [6] and if α is any positive integer then the method reduces to the N_{q_p}-method of Moursund [5].

Let $f(t)$ be a periodic function with period 2π and Lebesgue integrable over $(-\pi, \pi)$. Let

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \tag{5}$$

The series conjugate to (5) at $t = x$ is given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x). \tag{6}$$

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\}$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\}$$

$$\Psi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\psi(u)\mathrm{d}u \ (\alpha > 0)$$

$$\Psi_0(t) = \psi(t)$$

$$\psi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Psi_\alpha(t).$$

In his Ph D thesis [10] M Samal proved the following result:

Theorem M. *If $\phi_\alpha(t) \in BV(0,\pi)$, then the Fourier series of $f(t)$ is $|N_{q_\alpha}|$-summable at $t = x$ for $\alpha > 0$.*

2. Theorem and corollary

In the present paper we shall prove the following theorem.

Theorem. *If $\int_0^\pi \frac{|d\Psi_\alpha(t)|}{t^\alpha} < \infty$ and $\Psi_\alpha(+0) = 0$, then the series conjugate to the Fourier series of $f(t)$ is $|N_{q_\alpha}|$-summable at $t = x$ for $\alpha > 0$.*

By taking $q_\alpha(t) = \beta(1-t)^{\beta-1}$, where $0 < \alpha < \beta < h+1$, $[\alpha] = h$, in our theorem we obtain the following corollary.

COROLLARY [2]

If $\int_0^\pi \frac{|d\Psi_\alpha(t)|}{t^\alpha} < \infty$ and $\Psi_\alpha(+0) = 0$ then the series conjugate to the Fourier series of $f(t)$ at $t = x$ is summable $|C,\beta|$ for $\beta > \alpha > 0$.

3. Notations and lemmas

Notations

For our purpose we will use the following notations:

$$[w] = N$$

$$J(n,u) = \int_u^\pi (t-u)^{h-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}\cos nt\,\mathrm{d}t$$

$$S^{i,j}(x,u) = \sum_{n\le x}(x-n)^i\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^j\cos nu$$

$$\bar{S}^{i,j}(x,u) = \sum_{n\le x}(x-n)^i\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^j\sin nu$$

$$p(w,t) = \sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)\cos nt$$

$$p_1(w,t) = \sum_{n\le w-1} q_\alpha\left(\frac{n}{w}\right)\cos nt$$

$$p_2(w,t) = \sum_{n \le w-(\pi/u)} q_\alpha(n/w)\cos nt, \quad 0 < t \le u < \pi \text{ and } wu > \pi$$

$$p_3(w,t) = \sum_{[w-(\pi/u)]+1}^{[w-1]} q_\alpha(n/w)\cos nt, \quad 0 < t \le u < \pi \text{ and } wu > \pi.$$

Lemmas

The followings are some important lemmas on N_{q_α}-method.

Lemma 1. *The kernel $q_\alpha(t)$ is monotonic decreasing if $\alpha \ge 1$, its derivatives of odd orders less than h are negative and monotonic increasing, its derivatives of even orders less than h are positive and monotonic decreasing and there exists a constant A_h such that*

$$\left|\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} q_\alpha(t)\right| < A_h \quad (\beta = 0, 1, 2, 3, \ldots, h-1)$$

and

$$\int_0^1 \left|\frac{\mathrm{d}^h}{\mathrm{d}t^h} q_\alpha(t)\right| \mathrm{d}t < A_h.$$

This Lemma is an analogue of Lemma 2.2 of Moursund [5].

Lemma 2 [10]. *For $\alpha \ge 0, Q_h(t)$ is a continuous and monotonic increasing function of $t, Q(0) = 0, Q(1) = 1$ and $Q_h(t) \ge 0$.*

This follows directly from the definition of $Q(t)$ and $Q_h(t)$.

Lemma 3 [10].

$$\int_0^1 \frac{q^h(t)}{(1-t)^{\alpha-h}} \mathrm{d}t \text{ exists for all } \alpha \ge 0.$$

Proof. Let $\alpha > h$ (when $\alpha = h$ the result is valid in view of (iii), (iv) and (v). We have

$$\begin{aligned}\int_0^1 \frac{Q_h(t)}{t^{1+\alpha-h}} \mathrm{d}t &= \int_0^1 \frac{\mathrm{d}t}{t^{1+\alpha-h}} \int_{1-t}^1 q^h(u)\mathrm{d}u \\ &= \int_0^1 q^h(u)\mathrm{d}u \int_{1-u}^1 \frac{\mathrm{d}t}{t^{1+\alpha-h}} \\ &= -\frac{1}{\alpha-h}\left[\int_0^1 q^h(u)\mathrm{d}u - \int_0^1 \frac{q^h(u)}{(1-u)^{\alpha-h}}\mathrm{d}u\right].\end{aligned}$$

Since $\int_0^1 \frac{Q_h(t)}{t^{1+\alpha-h}}\mathrm{d}t$ and $\int_0^1 q^h(u)\mathrm{d}u$ exist, it follows that $\int_0^1 \frac{q^h(u)}{(1-u)^{\alpha-h}}\mathrm{d}u$ exists.

Lemma 4 [9]. *For $0 < \alpha < 1$, $\sum_{k=1}^\infty \frac{Q(1/k)}{K^{1-\alpha}}$ is convergent.*

Lemma 5 [1]. *If $\beta > \alpha > 0$, $\Psi_\alpha(t)$ is of $BV(0,\pi)$ and $\Psi_\alpha(+0) = 0$, then $\Psi_\beta(t)$ is an integral in $(0,\pi)$ and for almost all values of t,*

$$\Psi'_\beta(t) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^t (t-u)^{\beta-\alpha-1} \mathrm{d}\Psi_\alpha(u).$$

Lemma 6. *For* $0 < \alpha < 1$

$$\int_u^\pi (t-u)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} P_1(w,t)\mathrm{d}t = O(w^{\alpha+1}).$$

Proof.

$$\begin{aligned}
|p_1(w,t)| &\le \sum_{n<w-1} q_\alpha(n/w) \\
&< \int_0^w q_\alpha(x/w)\mathrm{d}x \\
&= w\int_0^1 q_\alpha(t)\mathrm{d}t \\
&= w.
\end{aligned}$$

Similarly $\left|\frac{\mathrm{d}}{\mathrm{d}t} p_1(w,t)\right| \le w^2$. Now

$$\begin{aligned}
\int_u^\pi (t-u)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} p_1(w,t)\mathrm{d}t &= \int_u^{u+(1/w)} (t-u)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} p_1(w,t)\mathrm{d}t \\
&\quad + \int_{u+(1/w)}^\pi (t-u)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} p_1(w,t)\mathrm{d}t \\
&= \int_u^{u+(1/w)} (t-u)^{-\alpha} O(w^2)\mathrm{d}t \\
&\quad + w^\alpha \int_{u+(1/w)}^\eta \frac{\mathrm{d}}{\mathrm{d}t} p_1(w,t)\mathrm{d}t,
\end{aligned}$$

for some $u + \frac{1}{w} < \eta < \pi$, by application of mean value theorem.

$$\begin{aligned}
&= O\left\{ w^2 \int_u^{u+(1/w)} (t-u)^{-\alpha}\mathrm{d}t \right\} + w^\alpha \left\{ p_1(w,\eta) - p_1\left(w, u+\frac{1}{w}\right)\right\} \\
&= O(w^{\alpha+1}) + O(w^{\alpha+1}) \\
&= O(w^{\alpha+1}).
\end{aligned}$$

Lemma 7. For $0 < \alpha < 1$, $0 < u < \pi$ *and* $wu > \pi$

$$\int_u^\pi (t-u)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} p_2(w,t)\mathrm{d}t = O\left\{ \frac{w^\alpha q_\alpha(1-(\pi/wu))}{u} \right\}.$$

Proof. We have

$$|p_2(w,t)| = \left| \sum_{n \le w(\pi/u)} q_\alpha(n/w) \cos nt \right|$$

$$\leq q_\alpha\left(1-\frac{\pi}{wu}\right)\max_{1<L,L'<w-(\pi/u)}\left|\sum_{L}^{L'}\cos nt\right|$$

$$<\frac{Kq_\alpha(1-(\pi/wu))}{t}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}p_2(w,t)\right|=\left|\sum_{n\leq w-(\pi/u)} nq_\alpha(n/w)\sin nt\right|$$

$$\leq\left(w-\frac{\pi}{u}\right)q_\alpha\left(1-\frac{\pi}{wu}\right)\max_{1<L,L'<w-(\pi/u)}\left|\sum_{L}^{L'}\sin nt\right|$$

$$<\frac{Kwq_\alpha(1-(\pi/wu))}{t}.$$

Now using the technique used in the second half of the proof of the Lemma 6 it can be proved that

$$\int_u^\pi (t-w)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_2(w,t)\mathrm{d}t=O\left\{\frac{w^\alpha q_\alpha(1-(\pi/wu))}{u}\right\}.$$

Lemma 8. *For* $0<\alpha<1$, $0<u<\pi$ *and* $wu>\pi$

$$\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_3(w,t)\mathrm{d}t=O\left\{w^{\alpha+1}Q\left(\frac{\pi}{wu}\right)\right\}.$$

Proof.

$$|P_3(w,t)|\leq\sum_{[w-(\pi/u]+1}^{[w-1]} q_\alpha(n/w)$$

$$<\int_{w-(\pi/u)}^{w} q_\alpha(x/w)\mathrm{d}x$$

$$=w\int_{1-(\pi/wu)}^{1} q_\alpha(t)\mathrm{d}t=wQ(\pi/wu).$$

Similarly, $|\frac{\mathrm{d}}{\mathrm{d}t}p_3(w,t)|<w^2Q(\pi/wu)$.

Using the technique similar to that used in the second half of the proof of the Lemma 6, it can be proved that

$$\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p_3(w,t)\mathrm{d}t=0\{w^{\alpha+1}Q(\pi/wu)\}.$$

Lemma 9. *For* $0<\alpha<1$,

$$\int_u^\pi (t-u)^{-\alpha}\sin nt\,\mathrm{d}t=O(n^{\alpha-1})$$

Proof. By application of mean value theorem for some $u+(1/n)<\eta<\pi$

$$\int_u^\pi (t-u)^{-\alpha}\sin nt\,\mathrm{d}t = \int_u^{u+(1/n)} (t-u)^{-\alpha}\sin nt\,\mathrm{d}t + \int_{u+(1/n)}^{\pi} (t-u)^{-\alpha}\sin nt\,\mathrm{d}t$$

$$= O\left\{\int_u^{u+(1/n)} (t-u)^{-\alpha}\mathrm{d}t\right\} + n^\alpha \int_{u+(1/n)}^{n} \sin nt\,\mathrm{d}t$$

$$= O(n^{\alpha-1}) + n^\alpha \frac{\cos u(n+(1/n)) - \cos n\eta}{n}$$

$$= O(n^{\alpha-1}).$$

Lemma 10. For $\alpha \geq 1$.

$$\sum_{n\leq w} q_\alpha\left(\frac{n}{w}\right) J(n,u) = O(w^{\alpha+1}).$$

Proof.

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^h p(w,t)\right| \leq w^h \sum_{n\leq w} q_\alpha\left(\frac{n}{w}\right)$$

$$\leq w^h\left\{q_\alpha(o) + \int_o^w q_\alpha\left(\frac{x}{w}\right)\mathrm{d}x\right\}$$

$$= w^h\left\{q_\alpha(o) + w\int_o^1 q_\alpha(t)\mathrm{d}t\right\}$$

$$= O(w^{h+1}), \text{ as } q_\alpha(o) \text{ is finite.}$$

Similarly, $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1} p(w,t) = O(w^{h+2})$.

Now proceeding as in the second half of the proof of the Lemma 6 it can be easily proved that

$$\sum_{n\leq w} q_\alpha\left(\frac{n}{w}\right) J(n,u) = O(w^{\alpha+1}).$$

Lemma 11 [3]. *Let $\lambda = \{\lambda_n\}$ be a positive monotonic increasing sequence with $\lambda_n \to \infty$ as $n \to \infty$.*

$$A_\lambda(x) = A_\lambda^o(x) = \sum_{\lambda_n\leq x} a_n$$

and

$$A_\lambda^r(x) = \sum_{\lambda_n\leq x} (x-\lambda_n)^r a_n \quad (r > 0).$$

Then if k is a positive integer,

$$A_\lambda(x) = \frac{1}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k A_\lambda^k(x).$$

Lemma 12 [[7], [8]]. *Let $C_n^{(i)}, S_n^{(i)}, \bar{S}_n^{(i)}$ denote the nth Cesaro sums of order $i \geq 0$ corresponding to the series $\sum_1^\infty (-1)^n n^j$, $\sum_1^\infty \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^j \cos nu$, $\sum_1^\infty (\mathrm{d}/\mathrm{d}u)^j \sin nu$ respectively.*

Then

(i) $$S_n^{(i)} = O(n^{i+j+1}), \quad 0 < u \leq \frac{1}{n}$$

(ii) $$S_n^{(i)} = O(n^j u^{-i-1}) + O(n^{i-1} u^{-j-1}), \quad \frac{1}{n} < u \leq \pi$$

(iii) $$\bar{S}_n^{(i)} = O(n^{i+j+1}), \quad O < u \leq \frac{1}{n}$$

(iv) $$\bar{S}_n^{(i)} = O(n^j u^{-i-1}) + O(n^i u^{-j-1}), \quad \frac{1}{n} < u \leq \pi$$

(v) $$C_n^{(i)} = O(n^{\max j, i-1)}) \quad \textit{if } j \textit{ is even and } \geq 2.$$

Lemma 13. *Let* $x > 0$,

(i) *If* $\frac{1}{x} < u \leq \pi$, *then*

$$S^{i,j}(x,u) = \begin{cases} O(x^i u^{-j-1}) & \textit{for} \quad 0 \leq j \leq i \\ O(x^j u^{-i-1}) & \textit{for} \quad j > i \geq 0. \end{cases}$$

(ii) *If* $\frac{1}{x} \geq u > 0$

$$S^{i,j}(x,u) = O(x^{i+j+1}).$$

(iii) *If* $\frac{1}{x} < u \leq \pi$, *then*

$$\bar{S}^{i,j}(x,u) = \begin{cases} O(x^i u^{-j-1}) & \textit{for} \quad 0 \leq j \leq i \\ O(x^j u^{-i-1}) & \textit{for} \quad j > i \geq 0. \end{cases}$$

Proof. If $[x] = m$, then by repeated application of Abel's transformation, we have

$$\begin{aligned}
S^{i,j}(x,u) &= \sum_{n \leq x} (x-n)^i \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^j \cos nu \\
&= \sum_{n=1}^{m-1} \Delta (x-n)^i S_n^{(o)} + (x-m)^i S_m^{(o)} \\
&= \sum_{n=1}^{m-2} \Delta^2 (x-n)^i S_n^{(1)} + [\Delta (x-n)^i]_{n=m-1} S_{m-1}^{(1)} + (x-m)^i S_m^{(o)} \\
&= \sum_{n=1}^{m-3} \Delta^3 (x-n)^i S_n^{(2)} + [\Delta^2 (x-n)^i]_{n=m-2} S_{m-2}^{(2)} \\
&+ [\Delta (x-n)^i]_{n=m-1} S_{m-1}^{(1)} + (x-m)^i S_m^{(o)} \\
&= \sum_{n=1}^{m-i} \Delta^i (x-n)^i S_n^{(i-1)} + \sum_{k=0}^{i-1} [\Delta^k (x-n)^i]_{n=m-k} S_{m-k}^{(k)}.
\end{aligned} \tag{7}$$

(i) Now for $k = 0, 1, 2, \ldots, (i-1)$ and $\frac{1}{x} < u \leq \pi$

$$[\Delta^k (x-n)^i]_{n=m-k} S_{m-k}^{(k)} = O(x^j u^{-k-1}) + O(x^{k-1} u^{-j-1}) \tag{8}$$

by Lemma 12 (ii).

Since $\Delta^i(x-n)^i$ is a constant $\sum_{n=1}^{m-i} \Delta^i(x-n)^i S_n^{(i-1)}$ is of order $S_n^{(i)}$. Thus using Lemma 12 (ii)

$$\sum_{n=1}^{m-i} \Delta^i(x-n)^i S_n^{(i-1)} = O(x^j u^{-i-1}) + O(x^{i-1} u^{-j-1}). \tag{9}$$

Since $x^j u^{-k-1}$ and $x^{k-1} u^{-j-1}$ are respectively dominated by $x^j u^{-i-1}$ and $x^{i-1} u^{-j-1}$, we obtain using (8) and (9) in (7)

$$S^{i,j}(x,u) = O(x^j u^{-i-1}) + O(x^{i-1} u^{-j-1}).$$

So for $\frac{1}{x} < u \leq \pi$,

$$S^{i,j}(x,u) = \begin{cases} O(x^i u^{-j-1}) & \textit{for} \quad 0 \leq j \leq i \\ O(x^j u^{-i-1}) & \textit{for} \quad j > i \geq 0. \end{cases}$$

(ii) Now for $k = 0, 1, 2, \ldots, (i-1)$ and $\frac{1}{x} \geq u\, 0$

$$[\Delta^k (x-n)^i]_{n=m-k} S_{m-k}^{(k)} = O(x^{j+k+1}) \tag{10}$$

by Lemma 12 (i).

Also using Lemma 12 (i)

$$\sum_{n=1}^{m-i} \Delta^i(x-n)^i S_n^{(i-1)} = O(x^{i+j+1}). \tag{11}$$

Since x^{i+j+1} dominates x^{j+k+1}, we obtain after using (10) and (11) in (7)

$$S^{i,j}(x,u) = O(x^{i+j+1}).$$

(iii) The proof of (iii) is similar to that of (i).

Lemma 14. *For* $0 < u \leq \pi$, $wu > 2\pi$ *and* $\alpha > 0$

$$\int_u^{\pi} (t-u)^{h-\alpha} \mathrm{d}t \int_1^{w-\pi/u} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x$$
$$= O\left\{\frac{w^{\alpha-h} q^h(1-(\pi/wu))}{u^{h+1}}\right\}.$$

Proof.

$$\int_u^{\pi} (t-u)^{h-\alpha} \mathrm{d}t \int_1^{w-(\pi/u)} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x$$
$$= \int_u^{u+(1/w)} (t-u)^{h-\alpha} \mathrm{d}t \int_1^{w-(\pi/u)} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x$$
$$+ \int_{u+(1/w)}^{\pi} (t-u)^{h-\alpha} \mathrm{d}t \int_1^{w-(\pi/u)} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x$$
$$= I_1 + I_2 \text{ say}.$$

As $q^h\left(\frac{x}{w}\right)$ is monotonic increasing in x by the mean value theorem for some $1<\xi< w-\frac{\pi}{u}$,

$$
\begin{aligned}
I_1 &= \int_u^{u+(1/w)} (t-u)^{h-\alpha}\mathrm{d}t \int_1^{w-(\pi/u)} (-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x\\
&= \int_u^{u+(1/w)} (t-u)^{h-\alpha}\mathrm{d}t\frac{1}{w^h}\\
&\quad\times \int_1^{w-(\pi/u)}\left[(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^h q_\alpha(\theta)\right]_{\theta=(x/w)}\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x\\
&= \int_u^{u+(1/w)} (t-u)^{h-\alpha}\mathrm{d}t\frac{1}{w^h}\int_1^{w-(\pi/u)} q^h\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x\\
&= w^{-h}\int_u^{u+(1+w)} (t-u)^{h-\alpha}\mathrm{d}t\, q^h\left(1-\frac{\pi}{wu}\right)\int_\xi^{w-(\pi/u)}\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x\\
&= w^{-h}q^h\left(1-\frac{\pi}{wu}\right)\int_u^{u+(1/w)}(t-u)^{h-\alpha}\left\{s^{h,h+1}\left(w-\frac{\pi}{u},t\right)-S^{h,h+1}(\xi,t)\right\}\mathrm{d}t\\
&= w^{-h}q^h\left(1+\frac{\pi}{wu}\right)\int_u^{u+(1/w)}(t-u)^{h-\alpha}O(w^{h+1}t^{-h-1})\mathrm{d}t,
\end{aligned}
$$

using Lemma 13(i), (ii)

$$
\begin{aligned}
&= O\left\{\frac{wq^h(1-(\pi/wu))}{u^{h+1}}\int_u^{u+(1/w)}(t-u)^{h-\alpha}\mathrm{d}t\right\}\\
&= O\frac{w^{\alpha-h}q^h(1-(\pi/wu))}{u^{h+1}}.
\end{aligned}
$$

For some $1<\zeta<w-\frac{\pi}{u}$, by application of mean value theorem

$$
\begin{aligned}
I_2 &= \int_{u+(1/w)}^{\pi}(t-u)^{h-\alpha}\mathrm{d}t\frac{1}{w^h}\\
&\quad\times\int_1^{w-(\pi/u)}\left[(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^h q_\alpha(\theta)\right]_{\theta=(x/w)}\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x\\
&= w^{-h}\int_{u+(1/w)}^{\pi}(t-u)^{h-\alpha}\mathrm{d}t\, q^h\left(1-\frac{\pi}{wu}\right)\int_\zeta^{w-(\pi/u)}\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x\\
&= w^{-h}q^h\left(1+\frac{\pi}{u}\right)\int_{u+(1/w)}^{\pi}(t-u)^{h-\alpha}\left\{S^{h,h+1}\left(w-\frac{\pi}{u},t\right)-S^{h,h+1}(\zeta,t)\right\}\mathrm{d}t\\
&= w^{-h}q^h\left(1-\frac{\pi}{u}\right)w^{\alpha-h}\int_{u+(1/w)}^{\eta}\left\{S^{h,h+1}\left(w-\frac{\pi}{u},t\right)-S^{h,h+1}(\zeta,t)\right\}\mathrm{d}t,
\end{aligned}
$$

(for some $u+\frac{1}{u}<\eta<\pi$, by application of mean value theorem)

$$
\begin{aligned}
&= w^{\alpha-2h}q^h\left(1-\frac{\pi}{wu}\right)\left\{S^{h,h}\left(w-\frac{\pi}{u},\eta\right)-S^{h,h}(\zeta,\eta)\right.\\
&\qquad\left.-S^{h,h}\left(w-\frac{\pi}{u},u+\frac{1}{w}\right)+S^{h,h}\left(\zeta,u+\frac{1}{w}\right)\right\}
\end{aligned}
$$

$$= w^{\alpha-2h} q^h \left(1 - \frac{\pi}{wu}\right) O(w^h u^{-h-1}), \text{ by Lemma 13(i) (ii)}$$

$$= O\left\{\frac{w^{\alpha-h} q^h (1-(\pi/wu))}{u^{h+1}}\right\}.$$

This completes the proof of Lemma 14.

Lemma 15.

$$\sum_{n\leq x}(x-n)^r(-1)^n n^r = O(x^r) \text{ and } \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n\leq x}(x-n)^r(-1)^n n^r\right) = O(x^r).$$

Proof. If r is even

$$\begin{aligned}\left|\sum_{n\leq x}(x-n)^r(-1)^n n^r\right| &= \left|\sum_{n\leq x}(x-n)^r\left[\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r \cos nu\right]_{u=\pi}\right| \\ &= [S^{r,r}(x,u)]_{u=\pi} \\ &= O(x^r) \text{ by Lemma 13 (i).}\end{aligned}$$

If r is odd

$$\begin{aligned}\left|\sum_{n\leq x}(x-n)^r(-1)^n n^r\right| &= \left|\left[\sum_{n\leq x}(x-n)^r\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r \sin nu\right]_{\pi/2}^{\pi}\right| \\ &= [S^{-r,r}(x,u)]_{u=\pi} \\ &= O(x^r) \text{ by Lemma 13 (iii).}\end{aligned}$$

Similarly it can be proved that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n\leq x}(x-n)^r(-1)^n n^r\right) = O(x^r).$$

Lemma 16. *For* $0 < u \leq \pi, wu > 2\pi$ *and* $\alpha > 0$

$$\int_u^{\pi}(t-u)^{h-\alpha}\mathrm{d}t\int_{w-(\pi/u)}^{w}(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}s^{h,h+1}(x,t)\mathrm{d}x$$

$$= O\left\{\frac{w^{\alpha-h+1}Q_h(\pi/wu)}{u^h}\right\}.$$

Proof.

$$\int_u^{\pi}(t-u)^{h-\alpha}\mathrm{d}t\int_{w-(\pi/u)}^{w}(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x$$

$$= \int_u^{u+(1/w)}(t-u)^{h-\alpha}\mathrm{d}t\int_{w-(\pi/u)}^{w}(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x$$

$$+\int_{u+(1/w)}^{\pi}(t-u)^{h-\alpha}\mathrm{d}t\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}(x/w)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x$$

$$=J_1+J_2 \text{ say.}$$

$$J_1=h\int_{u}^{u+(1/w)}(t-u)^{h-\alpha}\mathrm{d}t\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)S^{h-1,h+1}(x,t)\mathrm{d}x$$

$$=h\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)\mathrm{d}x\int_{u}^{u+(1/w)}(t-u)^{h-\alpha}S^{h-1,h+1}(x,t)\mathrm{d}t$$

$$=h\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}(x/w)\int_{u}^{u+(1/w)}(t-u)^{h-\alpha}O(x^{h+1}t^{-h})\,\mathrm{d}t,$$

by Lemma 13 (i)

$$=h\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)\mathrm{d}xO\left\{\frac{x^{h+1}}{u^{h}}\int_{u}^{u+(1/w)}(t-u)^{h-\alpha}\mathrm{d}t\right\}$$

$$=h\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)\mathrm{d}xO(u^{-h}x^{h+1}w^{\alpha-h-1})$$

$$=O\left\{u^{-h}w^{\alpha-h-1}\int_{w-(\pi/u)}^{w}\left[(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{h}q_{\alpha}(\theta)\right]_{\theta=x/w}x^{h+1}w^{-h}\,\mathrm{d}x\right\}$$

$$=O\left\{u^{-h}w^{\alpha-h}\int_{w-(\pi/u)}^{w}q^{h}(x/w)\mathrm{d}x\right\}$$

$$=O\left\{u^{-h}w^{\alpha-h+1}\int_{1-(\pi/wu)}^{1}q^{h}(\theta)\mathrm{d}\theta\right\}$$

$$=O\left\{u^{-h}w^{\alpha-h+1}Q_{h}\left(\frac{\pi}{wu}\right)\right\}$$

by application of mean value theorem for some $u+\frac{1}{w}<\zeta<\pi$,

$$J_2=h\int_{u+(1/w)}^{\pi}(t-u)^{h-\alpha}\mathrm{d}t\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)S^{h-1,h+1}(x,t)\mathrm{d}x$$

$$=h\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)\mathrm{d}x\int_{u+(1/w)}^{\pi}(t-u)^{h-\alpha}S^{h-1,h+1}(x,t)\mathrm{d}t$$

$$=h\int_{w-(\pi/u)}^{w}(-1)^{h}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h}q_{\alpha}\left(\frac{x}{w}\right)\mathrm{d}x\,w^{\alpha-h}\int_{u+(1/w)}^{\zeta}S^{h-1,h+1}(x,t)\mathrm{d}t$$

$$=hw^{\alpha-2h}\int_{w-(\pi/u)}^{w}q^{h}(x/w)O(x^{h}u^{-h})\,\mathrm{d}x,\ \text{by lemma 13 (i)}$$

$$=O\left(w^{\alpha-h+1}u^{-h}\int_{1-(\pi/wu)}^{1}q^{h}(\theta)\mathrm{d}\theta\right)$$

$$=O\left(\frac{w^{\alpha-h+1}Q_{h}(\pi/wu)}{u^{h}}\right).$$

Hence

$$\int_u^{\pi}(t-u)^{h-\alpha}\mathrm{d}t\int_{w-(\pi/u)}^{w}(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x$$

$$=O\left\{\frac{w^{\alpha-h+1}Q_h(\pi/wu)}{u^h}\right\}.$$

Lemma 17. *For $\alpha\geq 1$,*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}p(w,t)=\frac{1}{h!}\int_1^w(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x.$$

Proof.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}p(w,t)=\sum_{n\leq w}q_\alpha\left(\frac{n}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}\cos nt$$

$$=q_\alpha(1)\sum_{n\leq w}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}\cos nt-\int_1^w\frac{\mathrm{d}}{\mathrm{d}x}q_\alpha\left(\frac{x}{w}\right)\left\{\sum_{n\leq x}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}\cos nt\right\}\mathrm{d}x$$

$$=-\int_1^w\frac{\mathrm{d}}{\mathrm{d}x}q_\alpha\left(\frac{x}{w}\right)\left\{\sum_{n\leq x}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}\cos nt\right\}\mathrm{d}x,\ \text{since } q_\alpha(1)=0 \text{ for } \alpha\geq 1$$

$$=-\frac{1}{h!}\int_1^w\frac{\mathrm{d}}{\mathrm{d}x}q_\alpha\left(\frac{x}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h\left\{\sum_{n\leq x}(x-n)^h\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}\cos nt\right\}\mathrm{d}x,\ \text{by Lemma 11}$$

$$=-\frac{1}{h!}\int_1^w\frac{\mathrm{d}}{\mathrm{d}x}q_\alpha\left(\frac{x}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h S^{h,h+1}(x,t)\mathrm{d}x$$

$$=\frac{1}{h!}\left[\sum_{k=1}^{h-1}(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k q_\alpha\left(\frac{x}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{h-k}S^{h,h+1}(x,t)\right]_1^w$$

$$+\frac{1}{h!}\int_1^w(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x,$$

integrating by parts $(h-1)$ times,

$$=\frac{1}{h!}\int_1^w(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{h,h+1}(x,t)\mathrm{d}x,$$

as the integrated part vanishes for $x=1$ and $x=w$.

Lemma 18 [10]. *For $\alpha\geq 1$*

$$\sum_{n\leq w}(-1)^n n^h q_\alpha\left(\frac{n}{w}\right)=O\left\{q^h\left(1-\frac{1}{w}\right)\right\}+O\left\{wQ_h\left(\frac{1}{w}\right)\right\}.$$

Proof. Using the technique used in the proof of Lemma 17 we get

$$\sum_{n\le w}(-1)^n n^h q_\alpha(n/w)$$

$$= \frac{1}{h!}\int_1^w (-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n\le x}(x-n)^h(-1)^n n^h\right\}\mathrm{d}x$$

$$= \frac{1}{h!}\int_1^{w-1} (-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n\le x}(x-n)^h(-1)^n n^h\right\}\mathrm{d}x$$

$$+ \frac{1}{h!}\int_{w-1}^w (-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{\mathrm{d}}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n\le x}(x-n)^h(-1)^n n^h\right\}\mathrm{d}x$$

$$= I_1 + I_2 \text{ say.}$$

$$I_1 = \left[(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)\right]_{x=w-1}\int_\xi^{w-1}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n\le x}(x-n)^h(-1)^n n^h\right\}\mathrm{d}x$$

by mean value theorem, for $1<\zeta<w-1$,

$$= \frac{1}{w^h}q^h\left(1-\frac{1}{w}\right)\left[\sum_{n\le x}(x-n)^h(-1)^n n^h\right]_\zeta^{w-1}$$

$$= O\left\{q^h\left(1-\frac{1}{w}\right)\right\} \text{ by Lemma 15.}$$

$$I_2 = O\left(\int_{w-1}^w (-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right)x^h\mathrm{d}x\right), \text{ by Lemma 15}$$

$$= O\left(w\int_{1-(1/w)}^1 \theta^h(-1)^h\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^h q_\alpha(\theta)\mathrm{d}\theta\right)$$

$$= O\left(wQ_h\left(\frac{1}{w}\right)\right).$$

Hence

$$\sum_{n\le w}(-1)^n n^h q_\alpha\left(\frac{n}{w}\right) = O\left\{q^h\left(1-\frac{1}{w}\right)\right\} + O\left\{wQ_h\left(\frac{1}{w}\right)\right\}.$$

Lemma 19 [10]. *Let* $\alpha \ge 1$. *For* $r = 0,1,2,\ldots,h-1$

$$\sum_{n\le w}(-1)^n n^r q_\alpha\left(\frac{n}{w}\right) = O(1).$$

Proof.

$$\sum_{n\le w}(-1)^n n^r q_\alpha\left(\frac{n}{w}\right) = -\int_1^w \frac{\mathrm{d}}{\mathrm{d}x}q_\alpha\left(\frac{x}{w}\right)\left\{\sum_{n\le x}(-1)^n n^r\right\}\mathrm{d}x.$$

Proceeding as in Lemma 17, we get

$$\sum_{n\le w}(-1)^n n^r q_\alpha\left(\frac{n}{w}\right)$$

$$= \frac{(-1)^{r+1}}{r!}\int_1^w\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r+1} q_\alpha\left(\frac{x}{w}\right)\left\{\sum_{n\le x}(x-n)^r(-1)^n n^r\right\}\mathrm{d}x$$

$$= \int_1^w O\left\{(-1)^{r+1}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{r+1} q_\alpha\left(\frac{x}{w}\right)x^r\right\}\mathrm{d}x, \text{ by Lemma 15}$$

$$= O\left\{\int_0^1 \frac{x^r}{w^r}(-1)^{r+1}\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{r+1} q_\alpha(\theta)\mathrm{d}\theta\right\}$$

$$= O\left\{\int_0^1 (-1)^{r+1}\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{r+1} q_\alpha(\theta)\mathrm{d}\theta\right\}$$

$$= O(1), \text{ by Lemma 1.}$$

4. Proof of the theorem

We shall prove our theorem in two cases namely Case I for $O < \alpha < 1$ and Case II for $\alpha \ge 1$.

Case I. For $O < \alpha < 1$.

We know $B_n(x) = \frac{2}{\pi}\int_0^\pi \psi(t)\sin nt\,\mathrm{d}t$.

By the use of Lemma 5, we have

$$B_n(x) = \frac{2}{\pi}\int_0^\pi \sin nt\,\psi(t)\mathrm{d}t$$

$$= \frac{2}{\pi}\int_0^\pi \sin nt\,\mathrm{d}t\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-u)^{-\alpha}\mathrm{d}\Psi_\alpha(u)$$

$$= \frac{2}{\alpha\Gamma(1-\alpha)}\int_0^\pi \mathrm{d}\Psi_\alpha(u)\int_u^\pi (t-u)^{-\alpha}\sin nt\,\mathrm{d}t$$

By Definition 2 the series $\sum B_n(x) \in |N_{q_\alpha}|$ if and only if

$$I = \int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} nB_n(x)q_\alpha\left(\frac{n}{w}\right)\right| < \infty.$$

Now

$$I = \int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)\frac{2n}{\pi\Gamma(1-\alpha)}\int_0^\pi \mathrm{d}\Psi_\alpha(u)\int_u^\pi (t-u)^{-\alpha}\sin nt\,\mathrm{d}t\right|$$

$$\le k\int_0^\pi |\mathrm{d}\Psi_\alpha(u)|\int_1^\infty\left|\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p(w,t)\mathrm{d}t\right|\frac{\mathrm{d}w}{w^2}.$$

Since $\int_0^\pi \frac{|\mathrm{d}\Psi_\alpha(u)|}{u^\alpha} < \infty$ it is enough to show that

$$\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}p(w,t)\mathrm{d}t\right| = O\left(\frac{1}{u^\alpha}\right). \tag{12}$$

Now

$$\begin{aligned}
&\int_1^\infty \frac{dw}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{d}{dt}p(w,t)dt\right| \\
&= \int_1^\infty \frac{dw}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\left\{\frac{d}{dt}p_1(w,t) - q_\alpha\left(\frac{N}{w}\right)N\sin Nt\right\}dt\right| \\
&\leq \int_1^\infty \frac{dw}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{d}{dt}p_1(w,t)dt\right| \\
&\quad + \int_1^\infty \frac{dw}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}q_\alpha(N/w)N\sin Nt\, dt\right|.
\end{aligned} \tag{13}$$

And

$$\begin{aligned}
&\int_1^\infty \frac{dw}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{d}{dt}p_1(w,t)dt\right| \\
&= \int_1^{\pi/u} \frac{dw}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{d}{dt}p_1(w,t)dt\right| \\
&\quad + \int_{\pi/u}^\infty \frac{dw}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{d}{dt}p_1(w,t)dt\right| \\
&= J_1 + J_2 \text{ say.}
\end{aligned} \tag{14}$$

By the use of Lemma 6

$$\begin{aligned}
J_1 &= \int_1^{\pi/u} O(w^{\alpha+1})\frac{dw}{w^2} = O\left(\int_1^{\pi/u} w^{\alpha-1}dw\right) \\
&= O(1/u^\alpha)
\end{aligned} \tag{15}$$

$$\begin{aligned}
J_2 &= \int_{\pi/u}^\infty \frac{dw}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\left\{\frac{d}{dt}p_2(w,t) + \frac{d}{dt}p_3(w,t)\right\}dt\right| \\
&\leq \int_{\pi/u}^\infty \frac{dw}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{d}{dt}p_2(w,t)dt\right| \\
&\quad + \int_{\pi/u}^\infty \frac{dw}{w^2}\left|\int_u^\pi (t-u)^{-\alpha}\frac{d}{dt}p_3(w,t)dt\right| \\
&= J_{21} + J_{22} \text{ say.}
\end{aligned} \tag{16}$$

By the use of Lemma 7

$$\begin{aligned}
J_{21} &= \int_{\pi/u}^\infty \frac{dw}{w^2} O\left\{\frac{w^\alpha q_\alpha(1-(\pi/wu))}{u}\right\} \\
&= O\left\{\int_{\pi/u}^\infty \frac{w^\alpha q_\alpha(1-(\pi/wu))}{u}\frac{dw}{w^2}\right\} \\
&= O\left\{u^{-\alpha}\int_0^1 \frac{q_\alpha(t)}{(1-t)^\alpha}dt\right\} \\
&= O(u^{-\alpha}) \text{ by Lemma 3.}
\end{aligned} \tag{17}$$

By the use of Lemma-8

$$J_{22} = \int_{\pi/u}^{\infty} \frac{\mathrm{d}w}{w^2} O\left\{w^{\alpha+1} Q\left(\frac{\pi}{wu}\right)\right\}$$

$$= O\left\{\int_{\pi/u}^{\infty} Q\left(\frac{\pi}{wu}\right) w^{\alpha-1} \mathrm{d}w\right\}$$

$$= O\left\{u^{-\alpha} \int_0^1 \frac{Q(t)}{t^{1+\alpha}} \mathrm{d}t\right\}$$

$$= O(u^{-\alpha}) \text{ by (vii), since for } O < \alpha < 1, Q_h(t) = Q(t). \tag{18}$$

Now (14), (15), (16), (17) and (18) together imply

$$\int_1^{\infty} \frac{\mathrm{d}w}{w^2} \left| \int_u^{\pi} (t-u)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} p_1(w,t) \mathrm{d}t \right| = O(u^{-\alpha}). \tag{19}$$

By the use of Lemma 9

$$\int_1^{\infty} \frac{\mathrm{d}w}{w^2} \left| \int_u^{\pi} (t-u)^{-\alpha} N q_\alpha(N/w) \sin Nt \, \mathrm{d}t \right|$$

$$= \int_1^{\infty} \frac{\mathrm{d}w}{w^2} O\{N^\alpha q_\alpha(N/w)\}$$

$$= O\left\{\int_1^{\alpha} N^\alpha q_\alpha(N/w) \frac{\mathrm{d}w}{w^2}\right\}$$

$$= O\left\{\sum_{k=1}^{\infty} \int_k^{k+1} N^\alpha q_\alpha(N/w) \frac{\mathrm{d}w}{w^2}\right\}$$

$$= O\left\{\sum_{k=1}^{\infty} k^\alpha \int_k^{k+1} q_\alpha(k/w) \frac{\mathrm{d}w}{w^2}\right\} \quad \text{as} \quad k < w < k+1$$

$$= O\left\{\sum_{k=1}^{\infty} \frac{k^\alpha}{k} \int_{1-(1/k+1)}^{1} q_\alpha(\theta) \mathrm{d}\theta\right\}$$

$$= O\left\{\sum_{k=1}^{\infty} \frac{Q(1/k+1)}{k^{1-\alpha}}\right\}$$

$$= O\left\{\sum_{k=1}^{\infty} \frac{Q(1/k)}{k^{1-\alpha}}\right\}$$

$$= O(1) \text{ by Lemma 4.} \tag{20}$$

Now (19) and (20) together imply (12) and the theorem is proved for $O < \alpha < 1$.

Case II. For $\alpha \geq 1$.

Now

$$nB_n(x) = -\frac{2}{\pi} \int_0^{\pi} \psi(t) \frac{\mathrm{d}}{\mathrm{d}t} \cos nt \, \mathrm{d}t$$

$$= \frac{2}{\pi}\left[\sum_{k=1}^{h}(-1)^k \Psi_k(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \cos nt\right]_0^\pi$$

$$+\frac{2}{\pi}(-1)^{h+1}\int_0^\pi \Psi_h(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1} \cos nt\,\mathrm{d}t.$$

By the use of Lemma 5,

$$\int_0^\pi \Psi_h(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1} \cos nt\,\mathrm{d}t$$

$$= \frac{1}{\Gamma(1+h-\alpha)}\int_0^\pi \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1} \cos nt\,\mathrm{d}t \int_0^t (t-u)^{h-\alpha}\mathrm{d}\Psi_\alpha(x)$$

$$= \frac{1}{\Gamma(1+h-\alpha)}\int_0^\pi \mathrm{d}\Psi_\alpha(u)\int_u^\pi (t-u)^{h-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1} \cos nt\,\mathrm{d}t.$$

Hence

$$nB_n(x) = \frac{2}{\pi}\left[\sum_{k=1}^{h}(-1)^k \Psi_k(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \cos nt\right]_0^\pi$$

$$+\frac{2}{\pi\Gamma(1+h-\alpha)}(-1)^{h+1}\int_0^\pi \mathrm{d}\Psi_\alpha(u)\int_u^\pi (t-u)^{h-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1} \cos nt\,\mathrm{d}t.$$

Since for $k = 1, 2, \ldots, h$, $\Psi_k(+O) = O$, $\Psi_k(u) = 0(1)$ and $\int_0^\pi \frac{|\mathrm{d}\Psi_\alpha(u)|}{u^\alpha} < \infty$, for the proof of theorem it is enough to show that (a)

$$\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \cos nt\right]_{t=\pi}\right| < \infty,$$

for $k = 1, 2, 3, \ldots, h$. and (b)

$$\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)J(n,u)\right| = O\left(\frac{1}{u^\alpha}\right).$$

Proof of (a). $k = 1, 2, 3, \ldots, h$.

If k is odd, then $[(\frac{\mathrm{d}}{\mathrm{d}t})^k \cos nt]_{t=\pi} = 0$.

If k is even i.e. $k = 2m (m = 1, 2, 3, \ldots)$ then

$$\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \cos nt\right]_{t=\pi} = \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2m} \cos nt\right]_{t=\pi}$$

$$= (-1)^m n^k \cos n\pi$$

$$= (-1)^m (-1)^n n^k.$$

Hence for k is even and $k \le h - 1$

$$\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha(n/w)\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \cos nt\right]_{t=\pi}\right|$$
$$= \int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha(n/w)(-1)^n n^k\right|$$
$$= \int_1^\infty \frac{\mathrm{d}w}{w^2} O(1), \text{ by Lemma 19}$$
$$= O\left(\int_1^\infty \frac{\mathrm{d}w}{w^2}\right)$$
$$= O(1).$$

Also if h is even then

$$\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^h \cos nt\right]_{t=\pi}\right|$$
$$= \int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)(-1)^n n^h\right|$$
$$= \int_1^\infty \frac{\mathrm{d}w}{w^2} O\left\{\left(q^h\left(1-\frac{1}{w}\right)\right) + O\left(wQ_h\left(\frac{1}{w}\right)\right)\right\}, \text{ by Lemma 18}$$
$$= O\left(\int_1^\infty q^h\left(1-\frac{1}{w}\right)\frac{\mathrm{d}w}{w^2}\right) + O\left(\int_1^\infty Q_h\left(\frac{1}{w}\right)\frac{\mathrm{d}w}{w}\right)$$
$$= O\left(\int_0^1 q^h(t)\mathrm{d}t + O\left(\int_0^1 \frac{Q_h(t)}{t}\mathrm{d}t\right)\right)$$
$$= O(1). \quad \text{(This follows from (vii) and Lemma 3).}$$

Hence the proof of (a) is over.

Proof of (b).

$$\int_1^{2\pi/u} \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)J(n,u)\right|$$
$$= \int_1^{2\pi/u} \frac{\mathrm{d}w}{w^2} O(w^{\alpha+1}), \text{ by Lemma 10}$$
$$= O\left(\int_1^{2\pi/u} w^{\alpha-1}\mathrm{d}w\right)$$
$$= O(1/u^\alpha). \tag{21}$$

For $O < u \le \pi, wu > 2\pi$

$$\left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)J(n,u)\right|$$
$$= \left|\sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)\int_u^\pi (t-u)^{h-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1}\cos nt\,\mathrm{d}t\right|$$

$$= \left| \int_u^{\pi} (t-u)^{h-\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{h+1} p(w,t)\mathrm{d}t \right|$$

$$= \frac{1}{h!} \left| \int_u^{\pi} (t-u)^{h-\alpha} \mathrm{d}t \int_1^{w} (-1)^n \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x \right|,$$

by Lemma 17

$$\leq \frac{1}{h!} \left| \int_u^{\pi} (t-u)^{h-\alpha} \mathrm{d}t \int_1^{w-\pi/u} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x \right|$$

$$+ \frac{1}{h!} \left| \int_u^{\pi} (t-u)^{h-\alpha} \mathrm{d}t \int_{w-\pi/u}^{w} (-1)^h \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^h q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x} S^{h,h+1}(x,t)\mathrm{d}x \right|$$

$$= O\left(\frac{w^{\alpha-h} q^h(1-(\pi/wu))}{u^{h+1}}\right) + O\left(\frac{w^{\alpha-h+1} Q_h(\pi/wu)}{u^h}\right)$$

by the use of Lemma 14 and Lemma 16 respectively.

So

$$\int_{2\pi/u}^{\infty} \frac{\mathrm{d}w}{w^2} \left| \sum_{n\leq w} q_\alpha\left(\frac{n}{w}\right) J(n,u) \right|$$

$$= O\left(\int_{2\pi/u}^{\infty} \frac{w^{\alpha-h} q^h(1-(\pi/wu))}{u^{h+1}} \frac{\mathrm{d}w}{w^2}\right) + O\left(\int_{2\pi/u}^{\infty} \frac{w^{\alpha-h+1} Q_h(\pi/wu)}{u^h} \frac{\mathrm{d}w}{w^2}\right)$$

$$= O\left(\frac{1}{u^\alpha} \int_0^1 \frac{q^h(t)}{(1-t)^{\alpha-h}} \mathrm{d}t\right) + O\left(\frac{1}{u^\alpha} \int_0^1 \frac{Q_h(t)}{t^{1+\alpha-h}} \mathrm{d}t\right)$$

$$= O\left(\frac{1}{u^\alpha}\right), \text{ by Lemma 3 and (vii).} \tag{22}$$

Now (21) and (22) together imply

$$\int_1^{\infty} \frac{\mathrm{d}w}{w^2} \left| \sum_{n\leq w} q_\alpha\left(\frac{n}{w}\right) J(n,u) \right| = O\left(\frac{1}{u^\alpha}\right).$$

This completes the proof of (b) and hence the proof of the theorem is over.

Acknowledgements

The author expresses his gratefulness to Dr B K Ray and Dr M Samal for their kind help during the preparation of this paper.

The author is also grateful to the referee for his valuable suggestions and criticism which led to the improvement of this paper.

References

[1] Bosanquet L S, Some extensions of Young's criterion for the convergence of Fourier series, *Quart. J. Math.* Oxford **6** (1935) 113–123

[2] Bosanquet L S and Hyslop J M, On the absolute summability of the allied series of a Fourier series, *Math. Zeit.* **42** (1937) 489–512

[3] Hardy G H and Riesz M, *The general theory of Derichlet Series*, Cambridge (1915)

[4] Moursund A F, On a method of summation of Fourier series, *Ann. Math.* (2), **33** (1932) 773–784

[5] Moursund A F, On a method of summation of Fourier series, (2nd paper), *Ann. Math.* **34** (1933) 778–798

[6] Nevanlinna F, *Uber die summation der Fourier Schen Reihen und integrale overskit av Finska Vetensk* **64A**, No. 3, (1921–1922) 14

[7] Pati T, On the absolute Riesz summability of Fourier series and its conjugate series, *Trans. Amer. Math. Soc.* **76** (1954) 351–374

[8] Pati T, On the absolute Riesz summability of Fourier series and its conjugate series and their derived series, *Proc. Nat. Inst. Sci. India* **23**, No. 5 (1957) 354–369

[9] Samal M, On the absolute N_q-summability of some series associated with Fourier series, *J. Indian Math. Soc.* **50** (1986) 191–209

[10] Samal M, Summability of Fourier series, *Ph D thesis in Mathematics*, Utkal University (1991)

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 3, October 1998, pp. 273–282.

Multidimensional modified fractional calculus operators involving a general class of polynomials

S P GOYAL and TARIQ O SALIM

Department of Mathematics, University of Rajasthan, Jaipur 302 004, India

MS received 22 December 1997; revised 10 June 1998

Abstract. In the present work, we introduce and study essentially a class of multidimensional modified fractional calculus operators involving a general class of polynomials in the kernel. These operators are considered in the space of functions $M_\gamma(R^n_+)$. Some mapping properties and fractional differential formulas are obtained. Also images of some elementary and special functions are established.

Keywords and phrases. Fractional calculus operators; Mellin transform; general class of polynomials; H-function.

1. Introduction and preliminaries

Srivastava [6] introduced and studied a general class of polynomials which is defined by

$$S_N^M[x] = \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r} x^r, \quad N = 0, 1, \ldots, \tag{1}$$

where M is an arbitrary positive integer and the coefficients $A_{N,r} (N, r \geq 0)$ are arbitrary constants real or complex.

This general class of polynomials (1.1) unifies and extends a number of classical orthogonal polynomials such as Jacobi polynomials, Hermite polynomials, Laguerre polynomials, Gagenbauer polynomials, Bessel polynomials and several other classes of generalized hypergeometric polynomials.

Throughout this paper we use some notations. As usual R and C represent the fields of real and complex numbers respectively. R^n denotes the set of n-tuple real numbers, R^n_+ non-negative real numbers, and C^n complex numbers. For brevity, we write x^λ for the product $x_1 \cdots x_n$ and x^p for $x_1^p \cdots x_n^p$ with $x = (x_1, \ldots, x_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in C^n$ and $p \in C$. Further we write $|\lambda|$ for the sum $\lambda_1 + \cdots + \lambda_n$. By φ_+ we mean the positive part of a function φ defined by

$$\varphi_+(x) = \begin{cases} \varphi(x), & \text{if} \quad \varphi(x) > 0, \\ 0, & \text{if} \quad \varphi(x) \leq 0. \end{cases} \tag{1.2}$$

2. Modified fractional integrals

The multidimensional modified fractional integral operators $Y_{+;n}^{\mu;N,M}$ and $Y_{-;n}^{\mu;N,M}$ are defined as follows:

$$Y_{+;n}^{\mu;N,M}f(x) = \frac{1}{\Gamma(\mu+1)}\frac{\partial^n}{\partial x_1\ldots\partial x_n}\int_{R_+^n}\left[\min\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)-1\right]_+^{\mu} \times S_N^M\left[z\left\{\min\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)-1\right\}^a\right]f(t)\mathrm{d}t, \quad (2.1)$$

and

$$Y_{-;n}^{\mu;N,M}f(x) = \frac{(-1)^n}{\Gamma(\mu+1)}\frac{\partial^n}{\partial x_1\cdots\partial x_n}\int_{R_+^n}\left[1-\max\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)\right]_+^{\mu} \times S_N^M\left[z\left\{1-\max\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)\right\}^a\right]f(t)\mathrm{d}t, \quad (2.2)$$

for $Re(\mu) > 0$.

The fractional integral operators (2.1) and (2.2) have a large number of special cases due to the presence of the general class of polynomials in the kernels of the integrals. We mention below a few of them for the sake of illustration.

If in (2.1) and (2.2), we set $M = 1$, $N = 0$ and $A_{0,0} = 1$, then the general class of polynomials reduces to unity, and we get

$$Y_{+;n}^{\mu;0,1}f(x) = \frac{1}{\Gamma(\mu+1)}\frac{\partial^n}{\partial x_1\cdots\partial x_n}\int_{R_+^n}\left[\min\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)-1\right]_+^{\mu}f(t)\mathrm{d}t = X_+^{\mu}f(x), \quad (2.3)$$

and

$$Y_{-;n}^{\mu;0,1}f(x) = \frac{(-1)^n}{\Gamma(\mu+1)}\frac{\partial^n}{\partial x_1\cdots\partial x_n}\int_{R_+^n}\left[1-\max\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)\right]_+^{\mu}f(t)\mathrm{d}t = X_-^{\mu}f(x), \quad (2.4)$$

where X_+^{μ} and X_-^{μ} are the modified fractional integrals introduced by Brychkov *et al* [1, pp. 246–248] and studied by Tuan and Saigo [10], and Raina [3]. On the other hand, by expressing the general class of polynomials involved in (2.1) and (2.2) by its series form (1.1), and changing order of summation and integration, we get

$$Y_{-;n}^{\mu;N,M}f(x) = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}X_+^{\mu+ra}f(x), \quad (2.5)$$

and

$$Y_{-;n}^{\mu;N,M}f(x) = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}X_-^{\mu+ra}f(x). \quad (2.6)$$

Now, if we set $n = 1$ in (2.1) and (2.2) (or in (2.5) and (2.6)), we get

$$Y_{+;1}^{\mu;N,M}f(x) = \frac{1}{\Gamma(\mu+1)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\left(\frac{x}{t}-1\right)^{\mu}S_N^M\left[z\left(\frac{x}{t}-1\right)^a\right]f(t)\mathrm{d}t = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}I_+^{\mu+ra}x^{-\mu-ra}f(x) \quad (2.7)$$

and

$$Y_{-;1}^{\mu;N,M} f(x) = \frac{-1}{\Gamma(\mu+1)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^{\infty} \left(1-\frac{x}{t}\right)^{\mu} S_N^M \left[z\left(1-\frac{x}{t}\right)^a\right] f(t)\mathrm{d}t$$
$$= \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r} z^r \frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)} I_-^{\mu+ra} x^{-\mu-ra} f(x) \qquad (2.8)$$

where I_+^{μ} and I_-^{μ} are the well-known Riemann–Liouville and Weyl integral operators respectively.

Also, on setting $M=1, A_{N,r} = \binom{N+\lambda}{N} \frac{1}{(\lambda+1)_r}$ in (2.1) and (2.2), and using the known result [9, p. 101, equation (5.1.6)] therein, we get

$$Y_{+;n}^{\mu;N,1} f(x) = \frac{1}{\Gamma(\mu+1)} \frac{\partial^n}{\partial x_1 \cdots \partial x_n} \int_{R_+^n} \left[\min\left(\frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n}\right) - 1\right]_+^{\mu}$$
$$\times L_n^{(\lambda)}\left(z\left\{\min\left(\frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n}\right) - 1\right\}^a\right) f(t)\mathrm{d}t, \qquad (2.9)$$

and

$$Y_{-;n}^{\mu;N,1} f(x) = \frac{(-1)^n}{\Gamma(\mu+1)} \frac{\partial^n}{\partial x_1 \ldots \partial x_n} \int_{R_+^n} \left[1 - \max\left(\frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n}\right)\right]_+^{\mu}$$
$$\times L_n^{(\lambda)}\left(z\left\{1 - \max\left(\frac{x_1}{t_1}, \ldots, \frac{x_n}{t_n}\right)\right\}^a\right) f(t)\mathrm{d}t, \qquad (2.10)$$

for $Re(\mu) > 0$, where $L_N^{(\lambda)}(x)$ stands for generalized Laguerre polynomial [9, p. 101, equation (5.1.6)].

Now by dividing R_+^n for a fixed $x \in R_+^n$ into n subdomains with zero-measure intersection,

$$R_+^n = \bigcup_{k=1}^{n} \left\{ t \in R_+^n \,\middle|\, \frac{x_k}{t_k} \le \frac{x_j}{t_j},\ j = 1, \ldots, n;\ j \ne k \right\}, \qquad (2.11)$$

the multidimensional fractional integral operator $Y_{+;n}^{\mu,N,M}$ can be expressed as a finite sum of single integrals

$$Y_{+;n}^{\mu;N,M} f(x) = \frac{1}{\Gamma(\mu+1)} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left[x_k \int_0^1 (1-t)^{\mu} t^{n-\mu-1} \right.$$
$$\left. \times S_N^M \left[z\left(\frac{1}{t} - 1\right)^a \right] f(x_1 t, \ldots, x_n t)\mathrm{d}t \right]. \qquad (2.12)$$

Similarly by dividing R_+^n into subdomains with zero-measure intersection,

$$R_+^n = \bigcup_{k=1}^{n} \left\{ t \in R_+^n \,\middle|\, \frac{x_k}{t_k} \ge \frac{x_j}{t_j},\ j = 1, \ldots, n;\ j \ne k \right\}, \qquad (2.13)$$

we obtain

$$Y_{-;n}^{\mu;N,M} f(x) = \frac{-1}{\Gamma(\mu+1)} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left[x_k \int_1^{\infty} (t-1)^{\mu} t^{n-\mu-1} \right.$$
$$\left. \times S_N^M \left[z\left(1 - \frac{1}{t}\right)^a \right] f(x_1, t, \ldots, x_n t)\mathrm{d}t \right]. \qquad (2.14)$$

3. Modified fractional integrals of some special functions

We shall require a known result due to Raina [3] contained in the following:

Lemma If $s=(s_1,\ldots,s_n)\in C^n$, $h=(h_1,\ldots,h_n)\in R_+^n$ *and* $t^{(|s/h|)-1}f(t)\in L_1(R_+)$,

$$\int_{R_+^n} x^{s-1} f(\max(x^{h_1},\ldots,x^{h_n}))\mathrm{d}x = \frac{|s/h|}{s_1\ldots s_n} f^*(|s/h|) \tag{3.1}$$

$Re(s_j)>0(j=1,\ldots,n)$, *and*

$$\int_{R_+^n} x^{s-1} f(\min(x_1^{h_1},\ldots,x_n^{h_n}))\mathrm{d}x = \frac{(-1)^{n-1}|s/h|}{s_1\ldots s_n} f^*(|s/h|) \tag{3.2}$$

$Re(s_j)<0(j=1,\ldots,n)$

where $f^*(t)$ *denotes the one-dimensional Mellin transform of* $f(x)$.

Now we may calculate the modified fractional integrals of some elementary and special functions.

(i) Set $f(x)=x^d$ for $d\in C^n$. Making use of the known formulas [10, p. 257, equations (3.5) and (3.6)], we get

$$Y_{+;n}^{\mu;N,M}x^d = \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r}z^r \frac{\Gamma(\mu+ra+1)}{\Gamma(\mu+1)} \frac{\Gamma(n-\mu-ra+|d|)}{\Gamma(n+|d|)} x^d, \tag{3.3}$$

provided that $Re(d_j)>-1(j=1,\ldots,n)$ and $n+Re(|d|)>Re(\mu+ra)$ $(r=0,1,\ldots,[N/M])$, and

$$Y_{-;n}^{\mu;N,M}x^d = \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r}z^r \frac{\Gamma(\mu+ra+1)}{\Gamma(\mu+1)} \frac{\Gamma(1-n-|d|)}{\Gamma(1+\mu+ra-n-|d|)} x^d, \tag{3.4}$$

provied that $Re(d_j)<-1(j=1,\ldots,n)$ and $n+Re(|d|)<Re(\mu+ra)+1(r=0,1,\ldots,[N/M])$.

(ii) Set $f(x)=\frac{x^{-d-1}}{\Gamma(|d|+1)}[\min(x_1,\ldots,x_n)-1]_+^{|d|}$ for $d\in C^n$, then making use of the known formulas [10, p. 259, equations (3.9) & (3.10)], we get for $Re(|d|)>0$

$$Y_{+;n}^{\mu;N,M}\frac{x^{-d-1}}{\Gamma(|d|+1)}[\min(x_1,\ldots,x_n)-1]_+^{|d|} = \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r}z^r \times\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}\cdot\frac{x^{-d-1}[\min(x_1,\ldots,x_n)-1]_+^{\mu+ra+|d|}}{\Gamma(\mu+ra+|d|+1)} \tag{3.5}$$

and

$$Y_{-;n}^{\mu;N,M}\frac{x^{-d-1}}{\Gamma(|d|+1)}[1-\max(x_1,\ldots,x_n)]_+^{|d|} = \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r}z^r \times\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}\frac{x^{-d-1}[1-\max(x_1,\ldots,x_n)]_+^{\mu+ra+|d|}}{\Gamma(\mu+ra+|d|+1)} \tag{3.6}$$

(iii) Set $f(x) = x^{-1}H^{m_1,n_1}_{p_1,q_1}[\min(x_1^h,\ldots,x_n^h)]$, for $h \in R_+$, where

$$H^{m_1,n_1}_{p_1,q_1}[t] = H^{m_1,n_1}_{p_1,q_1}\left[t \,\middle|\, \begin{matrix}(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1}\end{matrix}\right] = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}\theta(\sigma)t^{-\sigma}\mathrm{d}\sigma \tag{3.7}$$

$$\theta(\sigma) = \frac{\prod_{j=1}^{m_1}\Gamma(b_j+B_j\sigma)\prod_{j=1}^{n_1}\Gamma(1-a_j-A_j\sigma)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-b_j-B_j\sigma)\prod_{j=n_1+1}^{p_1}\Gamma(a_j+A_j\sigma)} \tag{3.8}$$

is well-known Fox's H-function [2]. For details of this function, one can refer to Srivastava *et al* [7, Chap 2].

To evaluate the modified fractional integral of this function, we use the multi-dimensional Mellin inversion formula [1]:

$$f(x) = \frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty} f^*(s)x^{-s}\mathrm{d}s \quad \text{for} \quad f^*(s) = \int_{R_+^n} x^{s-1}f(x)\mathrm{d}x = M\{f(x)\},$$

where, and in what follows, the notation

$$\int_{(\gamma)-i\infty}^{(\gamma)+i\infty} \quad \text{means} \quad \int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\cdots\int_{\gamma_n-i\infty}^{\gamma_n+i\infty}, \qquad \text{for } \gamma = (\gamma_1,\ldots,\gamma_n)\in R^n.$$

Now, making use of (3.2), we get

$$Y^{\mu;N,M}_{+;n}x^{-1}H^{m_1,n_1}_{p_1,q_1}[\min(x_1^h,\ldots,x_n^h)] = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}$$

$$\times\frac{(-1)^{n-1}}{h(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\frac{\Gamma(-\mu-ra-|s|)}{\Gamma(-|s|)}H^*(s)\frac{|s|}{s_1\cdots s_n}x^{-s-1}\mathrm{d}s, \tag{3.9}$$

where $H^*(s) = \theta\left(\frac{|s|}{h}\right)$

Now, interpreting (3.9) as the H-function by using the known formula [3, p. 158, equation (3.7)], we get

$$Y^{\mu;N,M}_{+;n}x^{-1}H^{m_1,n_1}_{p_1,q_1}[\min(x_1^h,\ldots,x_n^h)] = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}x^{-1}$$

$$\times H^{m_1,n_1+1}_{p_1+1,q_1+1}\left[\min(x_1^h,\ldots,x_n^h)\,\middle|\,\begin{matrix}(1+\mu+ra,h),(a_j,A_j)_{1,p_1}\\(b_j,B_j)_{1,q_1},(1,h)\end{matrix}\right], \tag{3.10}$$

provided that $\gamma_j < 0(j=1,\ldots,n)$, $-|\gamma| > Re(\mu+Ra) > 0$ $(r=0,1,\ldots,[N/M])$; and

$$\text{(i)}\quad \Delta = \sum_{j=1}^{n_1}A_j - \sum_{j=n_1+1}^{p_1}A_j + \sum_{j=1}^{m_1}B_j - \sum_{j=m_1+1}^{q_1}B_j > 0, \quad \text{or}$$

$$\text{(ii)}\quad \Delta = 0,\quad Re\left(\sum_{j=1}^{p_1}a_j - \sum_{j=1}^{q_1}b_j\right) - \frac{p_1-q_1}{2} + |\gamma|\left(\sum_{j=1}^{p_1}A_j - \sum_{j=1}^{q_1}B_j\right) > 1;$$

$$Re(a_j) < 1 - \frac{|\gamma|}{h}A_j(j=1,\ldots,n_1),\quad Re(b_j) > -\frac{|\gamma|}{h}B_j(j=1,\ldots,m_1).$$

Similarly, we have

$$Y^{\mu;N,M}_{-;n}x^{-1}H^{m_1,n_1}_{p_1,q_1}[\max(x_1^h,\ldots,x_n^h)] = \sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}x^{-1}$$
$$\times H^{m_1+1,n_1}_{p_1+1,q_1+1}\left[\max(x_1^h,\ldots,x_n^h)\middle|\begin{matrix}(a_j,A_j)_{1,p_1},(1+\mu+ra,h)\\(1,h),(b_j,B_j)_{1,q_1}\end{matrix}\right], \tag{3.11}$$

provided that $\gamma_j > 0 (j=1,\ldots,n)$, $Re(\mu+ra)>0$, $(r=0,1,\ldots,[N/M])$; $Re(a_j)<1-\frac{|\gamma|}{h}A_j(j=1,\ldots,n_1)$, $Re(b_j)>-\frac{|\gamma|}{h}B_j(j=1,\ldots,m_1)$ and conditions (i) and (ii) stated with (3.10) hold.

Note that on setting $N=0$, $M=1$ and $A_{0,0}=1$ in pairs of equations (3.3) and (3.4) and (3.10) and (3.11), we get the results established by Tuan and Saigo [10, p. 257, equations (3.5) and (3.6)] and by Raina [3, pp. 158-159, equations (3.11) and (3.12)] respectively. Morever, if $A_j=1(j=1,\ldots,p_1)$, $B_j=1(j=1,\ldots,q_1)$ and $h=1$, then (3.10) and (3.11) yield the corresponding formulas for Meijer's G-function obtained by Tuan and Saigo [10, p. 259, equations (3.13) and (3.14)]. Further if we set $n=1$ in (3.10) and (3.11), then we get the formulas obtained by Raina and Koul [4, p. 99, equation (7)] and [5, p. 277, equation (2.5)] respectively.

4. Modified fractional operators on space $M_\gamma(R^n_+)$

Following [10], let $M_\gamma(R^n_+)$ denote the space of functions f which are defined on R^n_+, where $\gamma=(\gamma_1,\ldots,\gamma_n)\in R^n$. It is proved there that $f\in M_\gamma(R^n_+)$, if and only if, f can be represented as the inverse Mellin transform,

$$f(x)=\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty} f^*(s)x^{-s}\mathrm{d}s, \tag{4.1}$$

of a function $f^*(s)$ infinitely differentiable and with compact support on $((\gamma)-i\infty, (\gamma)+i\infty)$.

Theorem 1. *(a) Let $Re(\mu)>0$; $\gamma_j+Re(d_j)<1(j=1,\ldots,n)$; $Re(\mu+ra)+Re(|d|)+|\gamma|<n$ $(r=0,1,\ldots,[N/M])$ for $d\in C^n$ and $\gamma\in R^n$, then the operator $x^dY^{\mu;N,M}_{+;n}x^{-d}$ is a homeomorphism of the space, $M_\gamma(R^n_+)$ onto itself.*

Moreover, it can be written in the form

$$x^dY^{\mu,N,M}_{+;n}x^{-d}f(x)=\sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}$$
$$\cdot\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\frac{\Gamma(-\mu-ra-|d|-|s|+n)}{\Gamma(-|d|-|s|+n)}f^*(s)x^{-s}\mathrm{d}s \tag{4.2}$$

(b) Let $Re(\mu)>0$; $\gamma_j+Re(d_j)>0$ $(j=1,\ldots,n)$;

$1+Re(\mu+ra)+Re(|d|)+|\gamma|>n(r=0,1,\ldots,[N/M])$ for $d\in C^n$ and $\gamma\in R^n$, then the operator $x^dY^{\mu,N,M}_{-;n}x^{-d}$ is a homeomorphism of the space $M_\gamma(R^n_+)$ onto itself, and

$$x^dY^{N,M}_{-;n}x^{-d}f(x)=\sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}$$
$$\cdot\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\frac{\Gamma(1+|d|+|s|-n)}{\Gamma(1+\mu+ra+|d|+|s|-n)}f^*(s)x^{-s}\mathrm{d}s. \tag{4.3}$$

Proof. (a) Making use of (3.3) and (4.1), we get

$$x^d Y_{+;n}^{N,M} x^{-d} f(x) = x^d Y_{+;n}^{\mu,N,M} x^{-d} \cdot \frac{1}{(2\pi i)^n} \int_{(\gamma)-i\infty}^{(\gamma)+i\infty} f^*(s) x^{-s} \mathrm{d}s$$

$$= \frac{x^d}{(2\pi i)^n} \int_{(\gamma)-i\infty}^{(\gamma)+i\infty} f^*(s) Y_{+;n}^{\mu,N,M} x^{-s-d} \mathrm{d}s$$

$$= \frac{1}{(2\pi i)^n} \int_{(\gamma)-i\infty}^{(\gamma)+i\infty} f^*(s) \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r} z^r$$

$$\times \frac{\Gamma(\mu + ra + 1)}{\Gamma(\mu + 1)} \cdot \frac{\Gamma(-\mu - ra - |d| - |s| + n)}{\Gamma(-|d| - |s| + n)} x^{-s} \mathrm{d}s.$$

Changing the order of summation and integration in the above expression, we easily get (4.2). The interchange of order of integration is possible since $f^*(s)$ has a compact support. The function

$$\sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r} z^r \frac{\Gamma(ra + \mu + 1)}{\Gamma(\mu + 1)} \cdot \frac{\Gamma(-\mu - ra - |d| - |s| + n)}{\Gamma(-|d| - |s| + n)} f^*(s)$$

has compact support and is infinitely differentiable on $((\gamma) - i\infty, (\gamma) + i\infty)$ if so does $f^*(s)$. Hence $x^d Y_{+;n}^{\mu,N,M} x^{-d}$ belongs to $M_\gamma(R_+^n)$. The continuity of the mapping $f \to x^d Y_{+;n}^{\mu,N,M} x^{-d} f$ in $M_\gamma(R_+^n)$ is obvious.

Letting $d = 0$ in the theorem, we obtain

COROLLARY

(a) *Let* $Re(\mu) > 0$, $\gamma_j < 1 (j = 1, \ldots, n)$ *for* $\gamma \in R^n$; $Re(\mu + ra) + |\gamma| < n(r = 0, 1, \ldots, [N/M])$, *then* $Y_{+;n}^{\mu,N,M}$ *is a homeomorphism of* $M_\gamma(R_+^n)$ *onto itself, and*

$$Y_{+;n}^{\mu,N,M} f(x) = \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r} z^r \frac{\Gamma(ra + \mu + 1)}{\Gamma(\mu + 1)} \cdot \frac{1}{(2\pi i)^n}$$

$$\times \int_{(\gamma)-i\infty}^{(\gamma)+i\infty} \frac{\Gamma(-\mu - ra - |s| + n)}{\Gamma(-|s| + n)} f^*(s) x^{-s} \mathrm{d}s. \tag{4.4}$$

(b) *Let* $Re(\mu) > 0$, $\gamma_j > 0$ $(j = 1, \ldots, n)$ *for* $\gamma \in R^n$; $1 + Re(\mu + ra) + |\gamma| > n(r = 0, 1, \ldots, [N/M])$, *then* $Y_{-;n}^{\mu,N,M}$ *is a homeomorphism of* $M_r(R_+^n)$ *onto itself, and*

$$Y_{=;n}^{\mu,N,M} f(x) = \sum_{r=0}^{[N/M]} \frac{(-N)_{Mr}}{r!} A_{N,r} z^r \frac{\Gamma(ra + \mu + 1)}{\Gamma(\mu + 1)} \cdot \frac{1}{(2\pi i)^n}$$

$$\times \int_{(\gamma)-i\infty}^{(\gamma)+i\infty} \frac{\Gamma(1 + |s| - n)}{\Gamma(1 + \mu + ra + |s| - n)} f^*(s) X^{-s} \mathrm{d}s. \tag{4.5}$$

Note that if the general class of polynomials is reduced to unity by setting $N = 0$, $M = 1$ and $A_{0,0} = 1$ in (4.2) and (4.3) then we get the results established by Tuan and Saigo [10, pp. 262–263, equations (5.2) and (5.5)]

5. Modified fractional differentials

Since $Y_{+;n}^{\mu,N,m}$ is a homeomorphism of $M_\gamma(R_+^n)$ onto itself under the conditions stated with (4.4), then there exists its inverse operator which we shall define as $(Y_{+;n}^{\mu,N,M})^{-1}$. This operator is also a homeomorphism of $M_\gamma(R_+^n)$ onto itself and is defined by

$$(Y_{+;n}^{\mu,N,M})^{-1}f(x)=\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\left[\sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\,\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}\times\frac{\Gamma(-\mu-ra-|s|+n)}{\Gamma(-|s|+n)}\right]^{-1}f^*(s)x^{-s}\mathrm{d}s. \tag{5.1}$$

Similarly the inverse operator $(Y_{-;n}^{\mu,N,M})^{-1}$ is defined by

$$(Y_{-;n}^{\mu,N,M})^{-1}f(x)=\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\left[\sum_{r=0}^{[N/M]}\frac{(-N)_{Mr}}{r!}A_{N,r}z^r\times\frac{\Gamma(ra+\mu+1)}{\Gamma(\mu+1)}\cdot\frac{\Gamma(1+|s|-n)}{\Gamma(1+\mu+ra+|s|-n)}\right]^{-1}f^*(s)x^{-s}\mathrm{d}s. \tag{5.2}$$

It is easy to prove that

$$\begin{aligned}Y_{+;n}^{\mu,N,M}(Y_{+;n}^{\mu,N,M})^{-1}f(x)&=(Y_{+;n}^{\mu,N,M})^{-1}Y_{+;n}^{\mu,N,M}f(x)=f(x);\\ Y_{-;n}^{\mu,N,M}(Y_{-;n}^{\mu,N,M})^{-1}f(x)&=(Y_{-;n}^{\mu,N,M})^{-1}Y_{-;n}^{\mu,N,M}f(x)=f(x).\end{aligned} \tag{5.3}$$

Now the series involved in (5.1) and (5.2) can be reciprocated [8], and can be written as

$$(Y_{+;n}^{\mu,N,M})^{-1}f(x)=\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\Gamma(\mu+1)\Gamma(-|s|+n)\sum_{e=0}^{\infty}\beta_e z^e f^*(s)x^{-s}\mathrm{d}s, \tag{5.4}$$

$$(Y_{-;n}^{\mu,N,M})^{-1}f(x)=\frac{1}{(2\pi i)^n}\int_{(\gamma)-i\infty}^{(\gamma)+i\infty}\frac{\Gamma(\mu+1)}{\Gamma(1+|s|-n)}\sum_{e=0}^{\infty}\eta_e z^e f^*(s)x^{-s}\mathrm{d}s, \tag{5.5}$$

where β_e are given by the recursion formula

$$\beta_0\alpha_0=1,\quad \sum_{l=0}^{q}\beta_l\alpha_{q-l}=0,\quad q=1,2,3,\ldots;\quad \alpha_0\neq 0, \tag{5.6}$$

or explicitly by

$$\beta_e=(-1)^e(\alpha_0)^{-e-1}\det\begin{bmatrix}\alpha_1 & \alpha_0 & 0 & 0\ldots 0\\ \alpha_2 & \alpha_1 & \alpha_0 & 0\ldots 0\\ \vdots & \vdots & & \\ \alpha_e & \alpha_{e-1} & & \ldots\alpha_1\end{bmatrix}, \tag{5.7}$$

where

$$\alpha_r=\begin{cases}\dfrac{(-N)_{Mr}}{r!}A_{N,r}\Gamma(ra+\mu+1)\Gamma(-\mu-ra-|s|+n), & \text{if } 0\le r\le [N/M],\\ 0, & \text{if } r>[N/M],\end{cases} \tag{5.8}$$

and η_e are given by the recursion formula

$$\eta_0\delta_0 = 1, \quad \sum_{l=0}^{q} \eta_l \delta_{q-l} = 0, \quad q = 1,2,3,\ldots; \quad \delta_0 \neq 0 \tag{5.9}$$

or explicitly by

$$\eta_e = (-1)^e (\delta_0)^{-e-1} \det \begin{bmatrix} \delta_1 & \delta_0 & 0 & 0\ldots 0 \\ \delta_2 & \delta_1 & \delta_0 & 0\ldots 0 \\ \vdots & \vdots & & \\ \delta_e & \delta_{e-1} & & \ldots \delta_1 \end{bmatrix}, \tag{5.10}$$

where

$$\delta_r = \begin{cases} \dfrac{(-N)_{Mr}}{r!} A_{N,r} \dfrac{\Gamma(ra+\mu+1)}{\Gamma(1+\mu+ra+|s|+n)}, & \text{if } 0 \leq r \leq [N/M], \\ 0, & \text{if } r > [N/M]. \end{cases} \tag{5.11}$$

Now assuming that $\mu = k$, where k is a positive integer and setting $A_{0,0} = 1$ in (5.4) and (5.5), and making use of the known formulas [10, p. 266, equations (7.5) and (7.6)], we get

$$\begin{aligned}(Y_{+;n}^{k;N,M})^{-1} f(x) = {} & \prod_{j=1}^{k} \left(n - j + x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n} \right) f(x) \\ & + \frac{1}{(2\pi i)^n} \int_{(\gamma)-i\infty}^{(\gamma)+i\infty} k!\,\Gamma(-|s|+n) \sum_{e=1}^{\infty} \beta_e z^e f^*(s) x^{-s} \mathrm{d}s, \end{aligned} \tag{5.12}$$

and

$$\begin{aligned}(Y_{-;n}^{k;N,M})^{-1} f(x) = {} & (-1)^k \prod_{j=1}^{k} \left(n - j + x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n} \right) f(x) \\ & + \frac{1}{(2\pi i)^n} \int_{(\gamma)-i\infty}^{(\gamma)+i\infty} \frac{k!}{\Gamma(1+|s|-n)} \sum_{e=1}^{\infty} \eta_e z^e f^*(s) x^{-s} \mathrm{d}s. \end{aligned} \tag{5.13}$$

In the above results (5.12) and (5.13), if we set $N = 0$ and $M = 1$ then we obtain the results of Tuan and Saigo [10, p. 266, equations (7.5) and (7.6)] and Raina [3, p. 161, equations (5.5) and (5.6)]

Acknowledgement

The authors are grateful to the referee for making certain valuable suggestions. One of the authors (TOS) is thankful to Al-Azhar University of Gaza, Palestine for providing the necessary financial support to carry out this investigation.

References

[1] Brychkov Y A, Glaeske H J, Prudnikov A P and Tuan V K, *Multidimensional integral transformations* (1992) (Philadelphia-Reading-Paris-Montreux-Tokyo-Melbourne: Gordon and Breach)

[2] Fox C, The G and H-functions as symmetrical Fourier kernels. *Trans. Am. Math. Soc.* **98** (1961) 395–429

[3] Raina R K, A note on multidimensional modified fractional calculus operators, *Proc. Indian Acad. Sci (Math. Sci.)* **106** (1996) 155–162
[4] Raina R K and Koul C L, Fractional derivatives of the H-function, *Jñānabha* **7** (1977) 97–105
[5] Raina R K and Koul C L, On Weyl fractional Calculus and H-function transform, *Kyngpook Math. J.* **21** (1981) 275–279
[6] Srivastava H M, A contour integral involving Fox's H-function, *Indian J. Math.* **14** (1972) 1–6
[7] Srivastava H M, Gupta K C and Goyal S P, *The H-functions of one and two variables with applications* (1982) (New Delhi-Madras: South Asian Publ.)
[8] Srivastava H M, Koul C L and Raina R K, A class of convolution integral equations, *J. Math. Anal. Appl.* **108** (1985) 63–72
[9] Szego G *Orthogonal polynomials* 4th edn. (1975) (Providence Rhode Island: Am. Math. Soc. Colloq. Pub.), Vol. 23
[10] Tuan V K and Saiga M, Multidimensional modified fractional calculus operators, *Math. Nachr.* **161** (1993) 253–270

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 108, No. 3, October 1998, pp. 283–303.

Stinespring representability and Kadison's Schwarz inequality in non-unital Banach star algebras and applications

S J BHATT

Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 388 120, India

MS received 23 July 1997; revised 10 June 1998

Abstract. A completely positive operator valued linear map ϕ on a (not necessarily unital) Banach $*$-algebra with continuous involution admits minimal Stinespring dilation iff for some scalar $k > 0$, $\phi(x)^*\phi(x) \leq k\phi(x^*x)$ for all x iff ϕ is hermitian and satisfies Kadison's Schwarz inequality $\phi(h)^2 \leq k\phi(h^2)$ for all hermitian h iff ϕ extends as a completely positive map on the unitization A_e of A. A similar result holds for positive linear maps. These provide operator state analogues of the corresponding well-known results for representable positive functionals. Further, they are used to discuss (a) automatic Stinespring representability in Banach $*$-algebras, (b) operator valued analogue of Bochner–Weil–Raikov integral representation theorem, (c) operator valued analogue of the classical Bochner theorem in locally compact abelian group G, and (d) extendability of completely positive maps from $*$-subalgebras. Evans' result on Stinespring respresentability in the presence of bounded approximate identity (BAI) is deduced. A number of examples of Banach $*$-algebras without BAI are discussed to illustrate above results.

Keywords. Stinespring representability; completely positive map; Kadison's Schwarz inequality; automatic representability; positive definite functions on a group; Bochner theorem.

1. Introduction

Let $\phi : A \to B(H)$ be a completely positive linear map from, a not-necessarily unital, Banach $*$-algebra A to bounded linear operators on a Hilbert space H. Theorem 2.1 asserts that ϕ admits a minimal Stinespring dilation iff ϕ satisfies CP-Schwarz inequality $\phi(x)^*\phi(x) \leq k\phi(x^*x)$ $(x \in A)$ for some scalar $k > 0$ iff ϕ is hermitian and satisfies Kadison's Schwarz inequality $\phi(h)^2 \leq k\phi(h^2)$ $(h = h^* \in A)$ iff ϕ is extendable as a completely positive map on the unitization A_e. Theorem 2.2 is a positive map analogue of this. These operator state analogues of [4, Theorem 37.11; 25, Theorem 3.2] are proved in §2 using elementary properties of CP-maps, Jordan order structure in Banach $*$-algebras (which is distinct from usual order in non-C^*-situation), enveloping C^*-algebra $C^*(A)$ of A and by creating operator state analogue, within the formalism of Stinespring dilation, of the arguments in [24]. The results are applied in § 3 to a variety of situations. Corollary 3.1 contains several Cauchy–Schwarz inequalities for CP-maps, including a dilation-free proof of the CP-Schwarz inequality for a 2-positive map on a Banach $*$-algebra with BAI. A simple example shows that a positive map on a C^*-algebra need not satisfy CP-Schwarz inequality, though Kadison's Schawarz inequality does hold for such a map on a Banach $*$-algebra with BAI. This brings out an essential difference between these two

inequalities in the non-commutative case. Sufficient conditions for automatic Stinespring representability of CP-maps (as well as automatic representability of positive functionals) are developed in Corollary 3.2. This operator valued version of [4, Theorem 37.15] supplements automatic continuity phenomena even in scalar case. It is shown in Corollaries 3.6, 3.7 that a CP-map on a $*$-subalgebra B of a Banach $*$-algebra A having BAI extends as a CP-map on A iff it is Stinespring representable, provided (a) A is hermitian and B is closed; or (b) B is Banachable and $C^*(B) \to C^*(A)$ injectively; or (c) B is an abstract Segal algebra over A. It follows that if B has BAI and if (a) holds, then every CP-map on B extends to A. This supplements Arveson extension theorem [1]. Corollary 3.8 implies that an operator valued positive linear map ϕ on a commutative Banach $*$-algebra A is extendable to A_e iff ϕ is an integral with respect to a semi-spectral measure on the hermitian Gelfand space. This is an operator valued analogue of Bochner–Weil–Raikov integral representation theorem [11, ch. VI, Theorem 21.2, p. 492]. It follows (Corollary 3.8) that a weakly continuous operator valued function x on a locally compact abelian group G is positive definite iff x is an integral with respect to a semi-spectral measure on the dual group. This provides an operator valued version of the classical Bochner Theorem [15, Sec. 33] as well as a linear version of Stone–Naimark–Ambrose–Godament Theorem [20, ch. XV, Theorem 3.1, p. 489] occupying its proper place midway between the two. Finally, the abstract results are illustrated in several concrete Banach $*$-algebras like convolution algebras, the algebra $C^p(H)$ of von-Neumann Schatten class operators and the Hardy space $H^p(U)$ with the Hadamard product.

2. Stinespring representability

Let $(A, \| \,\|)$ be a complex Banach $*$-algebra, not necessarily having identity, assumed throughout satisfying $\|x^*\| = \|x\| (x \in A)$. Let $A_e = A \oplus \mathbb{C}$ be the Banach $*$-algebra obtained by adjoining identity to A, $\|x + \lambda 1\| = \|x\| + |\lambda| (x + \lambda 1 \in A_e)$. For a Hilbert space H, let $P(A,H)$ be the collection of all positive linear maps ϕ from A to the C^*-algebra $B(H)$ of all bounded linear operators on H, positive in the sense that $\phi(x^*x) \geq 0$ for all $x \in A$. For $n \in \mathbb{N}$, let $M_n(A) = A \otimes M_n(\mathbb{C})$ be the full matrix algebra over A, a Banach $*$-algebra with projective cross-norm $\|z\| = \inf\{\sum \|x_i\|\|y_i\| : z = \sum x_i \otimes y_i$ in $A \otimes M_n(\mathbb{C})\}$. Let $\phi_n = \phi \otimes id : M_n(A) \to M_n(B(H)) \subset B(H_n), H_n = \sum^{\oplus} H(n \text{ times})$, be $\phi_n([x_{ij}]) = [\phi(x_{ij})]$. Then ϕ is completely positive if each ϕ_n is positive. Let $CP(A,H) = \{\phi \in P(A,H) : \phi \text{ is completely positive}\}$. If every positive functional on A is continuous, then every $\phi \in P(A,H)$ is continuous by a closed graph argument.

DEFINITION

A map $\phi \in CP(A,H)$ is Stinespring representable if there exists a Hilbert space K, a $*$-homomorphism $\pi : A \to B(K)$ and a bounded linear operator $V : H \to K$ such that (i) $\phi(x) = V^*\pi(x)V (x \in A)$, and (ii) $K = [\pi(A)VH]$, the closed linear span of $\{\pi(x)V\xi : x \in A, \xi \in H\}$.

The arguments in part (iii) of [27, ch. IV, Theorem 3.6] show that the Stinespring representation $\{\pi, K, V\}$ of ϕ is unique up to unitary equivalence. By a classic theorem of Stinespring (in the unital case, and Lance in the non-unital case (see remarks in [9, p. 89])), every completely positive map ϕ on a C^*-algebra A is Stinespring representable [27, ch. 4, Theorem 3.6]. Evans showed that this also holds when A is a Banach $*$-algebra with BAI [9, Theorem 2.13]. In the absence of the requirement (ii) in the above definition,

it is shown in [10] that given a not-necessarily bounded completely positive map φ defined on a subspace of form N^*N of a C^*-algebra A with N a left ideal, a $*$-representation π of A can be constructed satisfying above (i).

Theorem 2.1. *Let $\phi \in CP(A,H)$. The following are equivalent.*

(1) *ϕ is Stinespring representable.*
(2) *There exists a scalar $k > 0$ such that $\phi(x)^*\phi(x) \leq k\phi(x^*x)$ for all x in A.*
(3) *ϕ is hermitian (i.e. $\phi(x^*) = \phi(x)^*$ for all x) and there exists a scalar $k > 0$ such that $\phi(h)^2 \leq k\phi(h^2)$ for all $h = h^*$ in A.*
(4) *ϕ is extendable to $\phi^e \in CP(A_e, H)$.*
(5) *ϕ is continuous in the Gelfand–Naimark pseudo-norm p_∞.*
(6) *There exists $\tilde{\phi} \in CP(C^*(A), H)$ such that $\phi = \tilde{\phi} \circ j$ where $j : A \to C^*(A)$ is $j(x) = x +$ srad A, srad $A (= \ker p_\infty)$ being the star radical of A.*

Further, if ϕ is Stinespring representable, then ϕ is continuous and $\phi(x)^*\phi(x) \leq \|\phi^e(1)\|$ $\phi(x^*x)$ for all x in A.

A completely positive map is an operator valued analogue of a positive linear functional. Then Stinespring construction $\{\pi, K, V\}$ corresponds to the GNS construction; and the above definition provides analogue of representability of positive functionals [4, Defn. 37.10, p. 199]. Thus Theorem 1 is a *CP*-analogue of [4, Theorem 37.11, p. 199]. Note that there is a gap in the proof [4, Theorem 37.11] which has been repaired in [18] and [24]. If A is a C^*-algebra, then any $\phi \in CP(A,H)$ satisfies the *CP*-Schwarz inequality $\phi(x)^*\phi(x) \leq \|\phi\|\phi(x^*x) (x \in A)$ [27, ch. IV, Corollary 3.8, p. 199]; whereas $\phi \in P(A,H)$ is known to satisfy Kadison's Schwarz inequality $\phi(h)^2 \leq \|\phi\|\phi(h^2) (h = h^* \in A)$ [17, Ex.10.5.9, p. 770]. Thus Theorem 1.1, part (3) iff (4) of which is a *CP*-analogue of [25, Theorem 3.2], shows that these inequalities are intimately connected with Stinespring dilation. The following positive map analogue of the above theorem further clarifies the role of Kadison's inequality in extendability.

Theorem 2.2. *Let $\phi \in P(A,H)$. The following are equivalent.*

(1) *ϕ is extendable to $\phi^e \in P(A_e, H)$.*
(2) *ϕ is hermitian and for some scalar $k > 0, \phi(h)^2 \leq k\phi(h^2)$ for all $h = h^*$ in A.*
(3) *ϕ is continuous in p_∞.*
(4) *There exists $\tilde{\phi} \in P(C^*(A), H)$ such that $\phi = \tilde{\phi} \circ j$.*

When ϕ is extendable, it is continuous and $\phi(h)^2 \leq \|\phi^e(1)\|\phi(h^2)$ hold for all $h = h^* \in A$.

Examples in [14, § 21.39, p. 332] show that, even in the scalar case, in the relevant inequalities in above theorems, $\|\phi^e(1)\|$ cannot be replaced by $\|\phi\|$ in general. When A has BAI, every positive functional on A is representable [21, Theorem 4.5.14, p. 219], [25, Theorem 3.2]; and hence continuous. The following, part (b) of which recaptures [9, Theorem 2.13], contains operator valued analogue.

COROLLARY 2.3

Let A have BAI (e_i) with $\|e_i\| \leq 1$.

(a) *Let $\phi \in P(A,H)$. Then ϕ is extendable and $\phi(h)^2 \leq \|\phi\|\phi(h^2)$ for all $h = h^*$ in A.*
(b) *Let $\phi \in CP(A,H)$. Then ϕ is Stinespring representable and $\phi(x)^*\phi(x) \leq \|\phi\|\phi(x^*x)$ for all x in A.*

The proofs are based on several auxiliary results some of which appears to be of independent interest. The positive elements of A is $A^+ = \{\sum x_i^* x_i : x_i \text{ in } A, \text{ finite sums}\}$. We write $x \geq 0$ for $x \in A^+$. An n-state on A is $f = [f_{ij}] \in M_n(A')(A' = \text{dual of } A)$ such that for each $x = [x_{ij}] \geq 0$ in $M_n(A)$, $[f_{ij}(x_{ij}] \geq 0$ in $M_n(\mathbb{C})$.

Lemma 2.4. (*Banach $*$-algebra analogues of* [27, *Ch.* 4, 3.1–3.4])

(a) *Let $x \in M_n(A)$. Then $x \geq 0$ iff x is a finite sum of elements of form $[x_i^* x_j]$ with $x_1, \ldots, x_n$ in A.*
(b) *Let $x = [x_{ij}] \geq 0$ in $M_n(A)$. Then for all $y_1, \ldots, y_n$ in A, $\sum_{ij} y_i^* x_{ij} y_j \geq 0$.*
(c) *Let B be a C^*-algebra. Then $\phi : A \to B$ is completely positive iff for each n, for all $x_1, \ldots, x_n$ in A and for all $y_1, \ldots, y_n$ in B, $\sum_{ij} y_i^* \phi(x_i^* x_j) y_j \geq 0$.*
(d) *Let B be an abelian C^*-algebra. Then every positive linear map $\phi : A \to B$ is completely positive.*
(e) (*Analogue of* [17, *Ex.* 11.5.21, p. 884]) *Let $\phi \in P(A, H)$ be continuous. Then $\phi \in CP(A, H)$ iff for each n and for each n-state $[f_{ij}]$ on $B(H)$, $[f_{ij} \circ \phi]$ is an n-state on A.*

Lemma 2.5. *Let $\phi \in P(A, H)$.*

(a) *$\phi(y^* x) = \phi(x^* y)^*$ holds for all x, y in A.*
(b) *Assume at least one of the following.*
    (i) *There exists $k > 0$ such that $\phi(x)^* \phi(x) \leq k\phi(x^* x)$ for all x.*
    (ii) *A has BAI.*

Then ϕ is continuous and extendable as a positive linear map on A_e.

Proof. Applying [4, Lemma 37.6, p. 147] to the functionals $f_\xi(x) = \langle \phi(x)\xi, \xi \rangle$, $\xi \in H$, (a) follows. Assume (b(i)). Then $|f_\xi(x)|^2 \leq \|\phi(x)\xi\|^2 \|\xi\|^2 = \|\xi\|^2 \langle \phi(x)^* \phi(x)\xi, \xi \rangle \leq \|\xi\|^2 k$ $\langle \phi(x^* x)\xi, \xi \rangle = k\|\xi\|^2 f_\xi(x^* x)$. Hence by [24,18], f_ξ is representable; hence is hermitian; which in turn implies that $f_\xi^e(x + \lambda 1) = f_\xi(x) + \lambda k\|\xi\|^2$ gives a positive linear extension to A_e (using, e.g., the arguments in [4,Theorem 37.11, p. 199] wherein hermiticity is implicitly used). Thus $\phi^e(x + \lambda 1) = \phi(x) + \lambda k 1$ gives the desired extension of ϕ. The continuity of ϕ follows from continuity of positive functionals on unital Banach $*$-algebras. Assume (b(ii)) with (e_i) a BAI for A. Then f_ξ and ϕ are continuous by [4, Theorem 37.15, p. 201]; and by Cauchy–Schwarz inequality, $|f_\xi(x)|^2 \leq (\overline{\lim}\, f_\xi(e_i^* e_i))\, f_\xi(x^* x) (x \in A)$; which gives the representability of f_ξ, and hence continuity and extendability of ϕ.

Lemma 2.6. *Let $\phi \in CP(A, H)$. Let x, y in A. Then $\phi(y^* x)^* \phi(y^* x) \leq \|\phi(y^* y)\| \phi(x^* x)$.*

Proof. Let $X = A \otimes H$. For x, y in $X, x = \sum x_i \otimes \xi_i, y = \sum y_j \otimes \eta_j$, let $\beta(x, y) = \sum \langle \phi(y_i^* x_j)\xi_j, \eta_i \rangle$. Then β defines a sesquilinear form on X; and $\beta(x, x) \geq 0$ for all x, as ϕ is completely positive. By Cauchy–Schwarz inequality [28, Theorem 1.4, p. 4], $|\beta(x, y)|^2 \leq \beta(x, x)\beta(y, y)$. Taking $x \otimes \xi$ and $y \otimes \eta$ in X, we get $\langle \phi(y^* x)^* \phi(y^* x)\, \xi, \xi \rangle = \|\phi(y^* x)\xi\|^2 = \sup\{|\langle \phi(y^* x)\xi, \eta \rangle|^2 : \|\eta\| \leq 1 \text{ in } H\} = \sup\{|\beta(x \otimes \xi, y \otimes \eta)|^2 : \|\eta\| \leq 1 \text{ in } H\} \leq \beta(x \otimes \xi, x \otimes \xi) \sup\{\beta(y \otimes \eta, y \otimes \eta) : \|\eta\| \leq 1\} = \sup\{\langle \phi(y^* y)\eta, \eta \rangle| : \|\eta\| \leq 1\} \langle \phi(x^* x)\xi, \xi \rangle = \|\phi(y^* y)\|$ $\langle \phi(x^* x)\xi, \xi \rangle$.

Lemma 2.7. *Let $\phi \in P(A, H)$ be such that there exists $\tilde{\phi} \in P(C^*(A), H)$ satisfying $\tilde{\phi} \circ j = \phi$. Then $\phi \in CP(A, H)$ iff $\tilde{\phi} \in CP(A, H)$.*

Proof. Since $\tilde{\phi}$ has to be continuous, ϕ is also continuous. Also, $\tilde{\phi} \circ j = \phi$ implies that $(\tilde{\phi})_n \circ (j \otimes id) = \phi_n$ for each n. Indeed, for any $[x_{ij}] \in M_n(A)$, $(\tilde{\phi})_n([j(x_{ij})]) = [\tilde{\phi} \circ j(x_{ij})] = \phi_n([x_{ij}])$. Since j is a $*$-homomorphism, ϕ_n is positive implies that $(\tilde{\phi})_n$ is positive on $M_n(j(A))$. As $M_n(j(A))$ is dense in $M_n(C^*(A))$, and as $(\tilde{\phi})_n$ is continuous on $M_n(C^*(A))$, it follows that $(\tilde{\phi})_n : M_n(C^*(A)) \to M_n(B(H))$ is positive if $\phi \in CP(A,H)$. The converse is similarily verified.

Lemma 2.8. *Let A have BAI (e_i). Let $\|e_i\| \leq 1$ for all i. Let $\phi \in P(A,H)$.*

(a) *There exists a unique $\tilde{\phi} \in P(C^*(A),H)$ such that $\phi = \tilde{\phi} \circ j$.*
(b) *ϕ is continuous, hermitian and satisfies $\phi(h)^2 \leq \|\phi\|\phi(h^2)(h^* = h \in A)$. Further $\|\phi\| = \sup \|\phi(e_i)\|$.*
(c) *Let A be abelian. Then $\phi \in CP(A,H)$ and $\phi(x)^*\phi(x) \leq \|\phi\|\phi(x^*x)$ for all $x \in A$.*
(d) *Let A be unital, $\|1\| = 1$. Then $\|\phi|_{H(A)}\| = \|\phi(1)\| = \|\phi\|$.*

Above (c) extends [27, ch. IV, Prop. 3.9, p. 199] to Banach $*$-algebras. Does it hold in the absence of BAI for continuous ϕ?

Proof. (a) The functional $f_\xi(x) = \langle\phi(x)\xi,\xi\rangle, \xi \in H$, is continuous [4, Theorem 37.15, p. 201]; and by [7, Prop. 2.7.5, p. 49], there exists a positive functional $\tilde{f}_\xi$ on $C^*(A)$ such that $f_\xi = \tilde{f}_\xi \circ j$. Define $\tilde{\phi} : A/\text{srad}\, A \to B(H)$ by $\tilde{\phi}(j(x)) = \phi(x)$. Then $\tilde{\phi}$ is well defined. Indeed, $x \in \text{srad}\, A$ implies that $f(x^*x) = 0$ for all positive functionals f on A [4, Theorem 40.9, p. 223]. As A has BAI, each such f is representable [21, Theorem 4.5.14, p. 219], hence [4, Theorem 37.11, p. 199] gives $f(x) = 0$. Thus $f_\xi(x) = 0$ for all ξ in H; and so the operator numerical range [5, § 9, p. 85] $W(\phi(x)) = 0$, so that $\phi(x) = 0$. Thus $\tilde{\phi}$ defines a positive map. Further, $\tilde{\phi}$ is $|\,|$ continuous, $|\,|$ denoting the norm on the C^*-algebra $C^*(A)$. Indeed, let $\xi \in H$ be such that $\|\xi\| \leq \|\phi\|^{-1/2}$. Then for all x in A, $|f_\xi(x)| = |\langle\phi(x)\xi,\xi\rangle| \leq \|\phi\|\|x\|\|\xi\|^2 \leq \|x\|$, hence $\|f_\xi\| \leq 1$. By the Cauchy–Schwarz inequality, $|\langle\tilde{\phi}(j(x))\xi,\xi\rangle| = |\langle\phi(x)\xi,\xi\rangle| = \lim|f_\xi(e_i x)| \leq \overline{\lim} f_\xi(e_i^* e_i)^{1/2} f_\xi(x^*x)^{1/2} \leq \overline{\lim}\|\tilde{f}_\xi\|\|e_i\||j(x)| \leq |j(x)|$. Hence for all ξ with $\|\xi\| = 1, |\langle\tilde{\phi}(j(x))\xi,\xi\rangle| \leq \|\phi\||j(x)|$; and so the numerical radius $\nu(\tilde{\phi}(j(x))) \leq \|\phi\|\,|j(x)|$. Since numerical radius is a norm equivalent to the given norm [5, Theorem 9.8, p. 86 and Theorem 4.1, p. 34], it follows that $\tilde{\phi}$ is continuous; and by extention, we get desired positive map $\tilde{\phi} : C^*(A) \to B(H)$.
(b) The inequality follows from (a) and Kadison's Schwarz inequality in C^*-algebra.
(c) Let A be abelian. Then so is $C^*(A)$. By [27, ch. IV, Prop. 3.9, p. 199], $\tilde{\phi}$ is completely positive. But then $C^*(M_n(A)) = C^*(A \otimes M_n(\mathbb{C})) = C^*(A \hat{\otimes}_\pi M_n(\mathbb{C})) = C^*(A)\hat{\otimes}_\nu C^*(M_n(\mathbb{C})) = C^*(A) \otimes_\nu M_n(\mathbb{C}) = M_n(C^*(A))$ shows that ϕ is also completely positive. By [27, ch. IV, Corollary 3.8, p. 199], for any $x \in A, \phi(x)^*\phi(x) = \tilde{\phi}(j(x))^*\tilde{\phi}(j(x)) \leq \|\tilde{\phi}\|\tilde{\phi}\,(j(x)^*j(x)) \leq \|\phi\|\phi(x^*x)(x \in A)$.
(d) For any $h = h^* \in A$, above (b) and Lemma 2.6(a) imply that $\|\phi(h)\|^2 = \|\phi(h)^2\| \leq \|\phi(1)\|\,\|\phi(h^2)\| \leq \|\phi(1)\|\,\|\phi|_{H(A)}\|\,\|h\|^2$, so that $\|\phi|_{H(A)}\| \leq \|\phi(1)\|$. Hence $\|\phi|_{H(A)}\| = \|\phi(1)\|$. Further, $j(1)$ being the identity of $C^*(A)$, [17, Ex. 10.5.10, p. 770] implies that $\|\tilde{\phi}\| = \|\tilde{\phi}(j(1))\| = \|\phi(1)\|$. The conclusion follows from $\|\phi\| \leq \|\tilde{\phi}\|$.

Lemma 2.9. *Let $\phi \in CP(A,H)$ be Stinespring representable having the Stinespring representation $\{\pi, K, V\}$. Then ϕ is continuous and $\|\phi\| \leq \|V\|^2$. If A has BAI (e_i), $\|e_i\| \leq 1$, then $\|\phi\| = \|V\|^2$.*

Proof. Let $\phi(x)=V^*\pi(x)V$. Then $\|\phi(x)\|\leq\|V\|^2\|\pi(x)\|\leq\|V\|^2\|x\|$; hence $\|\phi\|\leq\|V\|^2$. If (e_i) is a BAI for A, then $\pi(e_i)\to 1$ strongly. Hence, for all $\xi\in H$, $\|V\xi\|^2=\langle V\xi,V\xi\rangle \leq\|\xi\|\,\|V^*V\xi\|=\|\xi\|\lim\|V^*\pi(e_i)V\xi\|=\|\xi\|\lim\|\phi(e_i)\xi\|\leq\|\xi\|^2\|\phi\|$; hence $\|V\|^2\leq\|\phi\|$.

Lemma 2.10. *$C^*(A_e)=(C^*(A))_e$ up to isometric $*$-isomorphism.*

It is interesting to note that this does not mean that A has identity iff $C^*(A)$ has identity. In fact, [2] contains an example of a non-unital Banach $*$-algebra A such that $C^*(A)$ has identity.

Proof. As noted in [25, p. 145], Gelfand–Naimark pseudonorm p_∞^A on $A=p_\infty^{A_e}|A=$ restriction to A of the Gelfand–Naimark pseudonorm on A_e. Since each $*$-representation of A can be extended to A_e, $(\text{srad } A_e)\cap A=\text{srad } A$. Note that $(x,\lambda)\in\text{srad } A_e$ iff $x\in\text{srad } A$ and $\lambda=0$. The map $\phi: A_e/\text{srad } A_e\to(A/\text{srad } A)_e$, $\phi((x,\lambda)+\text{srad } A_e)=(x+\text{srad } A)+\lambda 1=(x+\text{srad } A,\lambda)$ is a bijective $*$-isomorphism, hence extends to the desired isometric $*$-isomorphism between $C^*(A_e)$ and $(C^*(A))_e$.

A positive linear functional f on A is representable [4,21] if there exists a cyclic representation π of A into bounded linear operators on a Hilbert space such that $f(x)=\langle\pi(x)\xi,\xi\rangle(x\in A)$, where ξ is a cyclic vector for π.

Lemma 2.11. *Let f be a positive functional on A.*

(a) *f is representable iff for some $k>0$, $|f(x)|\leq kp_\infty(x)$ for all x.*
(b) (i) [25, *Lemma* 1.31 $|f(y^*xy)|\leq p_\infty(x)f(y^*y)(x,y$ *in* $A)$.
 (ii) *Let A be unital. Then $|f(x)|\leq f(1)p_\infty(x)(x\in A)$.*
 (iii) *Let A has BAI (e_i). Then $|f(x)|\leq(\overline{\lim} f(e_i^*e_i))p_\infty(x)(x\in A)$.*

A linear map $\phi: A\to B(H)$ is J-positive if $\phi(h^2)\geq 0$ for all $h=h^*\in A$. If each $\phi_n=\phi\otimes id$ on $M_n(A)$ is J-positive, then ϕ is completely J-positive. Note that if A is a C^*-algebra, then ϕ is positive iff ϕ is J-positive.

Lemma 2.12. *Let $\phi: A\to B(H)$ be linear satisfying the following.*

(i) *ϕ is hermitian.*
(ii) *ϕ is J-positive.*
(iii) *There exists a scalar $k>0$ such that $\phi(h)^2\leq k\phi(h^2)(h=h^*\in A)$. Then $\phi^e(x+\lambda 1)=\phi(x)+\lambda k1$ gives a J-positive extension of ϕ.*

Proof. Let $u=h+\lambda 1=u^*\in A_e$. Then $h=h^*$, $\lambda=\lambda^-$. For all $\xi\in H$,

$$\begin{aligned}\langle\phi^e(u^2)\xi,\xi\rangle &= \langle\phi(h^2)\xi,\xi\rangle+2\lambda\langle\phi(h)\xi,\xi\rangle+\lambda^2k\|\xi\|^2\\ &\leq\langle\phi(h^2)\xi,\xi\rangle-2|\lambda|\,|\langle\phi(h)\xi,\xi\rangle|+\lambda^2k\|\xi\|^2\\ &\leq\langle\phi(h)^2\xi,\xi\rangle/k-2|\lambda|\|\phi(h)\xi\|\|\xi\|+\lambda^2k\|\xi\|^2\\ &=[\,\|\phi(h)\xi\|/k^{1/2}-k^{1/2}|\lambda|\|\xi\|\,]^2\geq 0.\end{aligned}$$

Lemma 2.13. *Assume that A is symmetric and $\phi: A\to B(H)$ is linear, J-positive. Then ϕ is positive.*

By [21, Corollary 4.7.8, p. 233], every J-positive functional on a symmetric Banach $*$-algebra is positive. The conclusion follows by applying this to $f_\xi(x)=\langle\phi(x)\xi,\xi\rangle$, $\xi\in H$.

Lemma 2.14. [25] *Let $\Im$ denote the collection of all continuous positive linear functionals f on A such that $\|f|_{H(A)}\| \leq 1$. Let $\lambda(x) = \sup\{f(x^*x)^{1/2} : f \in \mathcal{F}\}, \tau(x) = \max\{\lambda(x), \lambda(x^*)\}$. The following statements hold.*

(a) *λ and τ are submultiplicative seminorms on A satisfying $\lambda(xy) \leq p_\infty(x)\lambda(y), p_\infty(x) \leq \lambda(x) \leq \tau(x), \tau(x) = \tau(x^*)$ for all x, y in A.*
(b) *If A is $*$-semisimple, then τ is a norm.*
(c) *If τ is a norm, then the τ-completion $\bar{A}$ of A is symmetric and $\tau(h^2) \leq p_\infty(h)\tau(h)$.*

Lemma 2.15. *Let $\phi \in P(A, H)$ satisfy statement* (2) *of Theorem* 2.2. *Then ϕ is extendable and $\phi^e(x + \lambda 1) = \phi(x) + \lambda k1$ defines a positive linear extension of ϕ to A_e.*

Proof. The positive functionals $f_\xi(x) = \langle \phi(x)\xi, \xi\rangle$, $\xi \in H$, are hermitian and satisfy $|f_\xi(h)|^2 \leq k\|\xi\|^2 f_\xi(h^2)(h = h^* \in A)$. By [25, Lemma 3.2], they are representable. By Lemma 2.11, $f_\xi(x) = 0$ for all ξ, all $x \in$ srad A. This implies that ϕ vanishes on the star radical srad A. Indeed, for any $h = h^*$ in srad A, $h^2 \in$ srad A and $f_\xi(h^2) = 0$, hence $\langle \phi(h)^2\xi, \xi\rangle = 0$ for all ξ, showing that $\phi(h) = 0, \phi(\text{srad } A) = \{0\}$. It follows that ϕ factors through the $*$-semisimple Banach $*$-algebra $A/\text{srad } A$ with the quotient norm; and continues to be positive and satisfying statement (2) of Theorem 2.2. Thus we can assume that A is $*$-semisimple.

Next we show that ϕ is τ-continuous on A. Let $\xi \in H$, $\mu = k\|\xi\|^2$. For all $h = h^* \in A$, one has $|f_\xi(h)|^2 \leq \mu^2 r(h^2)$, r denoting the spectral radius in A. Indeed, $|f_\xi(h)|^2 = |\langle \phi(h)\xi, \xi\rangle|^2 \leq \|\phi(h)\xi\|^2 = \|\xi\|^2\langle \phi(h)^2\xi, \xi\rangle \leq k\|\xi\|^2\langle \phi(h^2)\xi, \xi\rangle = k\|\xi\|^2 f_\xi(h^2)(h = h^* \in A)$. By [25, Theorem 3.2], f_ξ is representable, hence extendable to A_e, with $f_\xi(e) = k\|\xi\|^2$ $(e = (0,1)$, the identity in $A_e)$. Then [4, Lemma 37.6 (iii), p. 197] applied to f_ξ gives $f_\xi(h^2) = f_\xi(eh^2e) \leq f_\xi(e)r(h^2) = \mu r(h^2)$. It follows that $|f_\xi(h)|^2 \leq \mu^2 r(h^2)$. Thus $f_\xi/\mu \in \Im$ and $|f_\xi(h)| \leq \mu\tau(h)$ for all $h = h^*$. Then $\|\phi(h)\xi\|^2 = \langle \phi(h)^2\xi, \xi\rangle \leq k\langle \phi(h^2)\xi, \xi\rangle = kf_\xi(h^2) \leq k^2\|\xi\|^2\tau(h^2) \leq k^2\|\xi\|^2\tau(h)^2$. This ϕ is τ-continuous on $H(A)$, hence also on A.

Now the τ-completion $\bar{A}$ of A is symmetric by Lemma 2.14 (c); and by continuity, ϕ extends to a hermitian positive linear map $\bar{\phi} : \bar{A} \to B(H)$ satisfying statement (2) of Theorem 2.2 on $\bar{A}$. By Lemma 2.12, $\bar{\phi}^e : (\bar{A})_e \to B(H), \bar{\phi}^e(x + \lambda 1) = \bar{\phi}(x) + \lambda k1$ provides a J-positive extension of $\bar{\phi}$ to $\bar{A}_e$. But as $\bar{A}$ is symmetric, $\bar{A}_e$ is also symmetric [21, Theorem 4.7.9]. Lemma 2.13 implies that $\bar{\phi}^e$ is positive on $\bar{A}_e$. It follows that ϕ^e is positive on A_e.

Lemma 2.16. *Let $\phi \in CP(A, H)$.*

(a) *Statement* (2) *of Theorem* 2.1 *is equivalent to:*
$(2°)\phi_n(x)^*\phi_n(x) \leq k\phi_n(x^*x)$ for all $x \in M_n(A)$, all $n \in \mathbb{N}$.
(b) *Statement* (3) *of Theorem* 2.1 *is equivalent to:*
$(3°)$ ϕ is hermitian and $\phi_n(h)^2 \leq k\phi_n(h^2)$ for all $h = h^* \in M_n(A)$, all $n \in \mathbb{N}$.

In the above $(2°)$ and $(3°)$, the scalar $k > 0$ is same as in the corresponding statements in Theorem 2.1.

Proof. (a) Assume $\phi(x)^*\phi(x) \leq k\phi(x^*x)(x \in A)$. By Lemma 2.5 (b), ϕ is hermitian and extends to the positive linear map $\phi^e : A_e \to B(H), \phi^e(x + \lambda 1) = \phi(x) + \lambda 1$. By Lemma 2.8, there exists $(\phi^e)^\sim$ in $P(C^*(A_e), H)$ such that $(\phi^e) \circ j^e = \phi^e$ where $j^e : A_e \to A_e/\text{srad } A_e$ is the natural quotient map. In view of Lemma 2.10, $C^*(A_e) = (C^*(A))_e, j^e|_A = j$, srad $A = A \cap$ srad $A_e, p_\infty^A = p_\infty^{A_e}|_A$. Hence $\tilde{\phi}$ given by $\tilde{\phi} = (\phi^e)^\sim|_{C^*(A)} \in P(C^*(A), H)$ such

that $\tilde{\phi} \circ J = \phi$. Now by Lemma 2.7, $\tilde{\phi} \in CP(C^*(A), H)$. Thus $\tilde{\phi}$ is Stinespring representable, and $\tilde{\phi}(x) = V^*\pi(x)V(x \in C^*(A))$ (in the notations of the definition). Let $n \in \mathbb{N}, V_n = V \otimes id : H \otimes \mathbb{C}^n \to K \otimes \mathbb{C}^n, \pi_n = \pi \otimes id : M_n(C^*(A)) \to M_n(B(H))$ a $*$-homomorphism. Then $(\tilde{\phi})_n = \phi \otimes id = (V^*\pi(\ )V) \otimes id = (V \otimes id)^*(\pi \otimes id)(V \otimes id) = V_n^*\pi_n(\ )V_n$. Hence for all x in $M_n(C^*(A))$, $\tilde{\phi}_n(x)^*\tilde{\phi}_n(x) = V_n^* \ \pi_n(x)^*V_nV_n^*\pi_n(x)V_n \leq \|V_n^*V_n\|V_n^*\pi_n(x^*x)V_n = \|V_n\|^2\tilde{\phi}_n(x^*x) = \|V\|^2\tilde{\phi}_n(x^*x) = \|\tilde{\phi}\|\tilde{\phi}_n(x^*x)$ using Lemma 2.9. (Alternatively, each $\tilde{\phi}_n$ is CP on $M_n(C^*(A))$, hence for all x in $M_n(C^*(A))$, $\tilde{\phi}_n(x)^*\tilde{\phi}_n(x) \leq \|\tilde{\phi}_n\|\tilde{\phi}_n(x^*x) = \|\tilde{\phi}\|\tilde{\phi}_n(x^*x)$. As $\tilde{\phi} \circ j = \phi, (\tilde{\phi})_n \circ (j \otimes id) = \phi_n$, we obtain $\phi_n(x)^*\phi_n(x) \leq \|\tilde{\phi}\|\phi_n(x^*x)$ $(x \in M_n(A))$, and $\|\tilde{\phi}\| \leq \|(\phi^e)^\sim\| = \|(\phi^e)^\sim(1)\| = \|\phi^e(1)\| = k$.)
(b) This can be proved using Lemma 2.15 instead of Lemma 2.5 (b).

Lemma 2.17. *Let $\phi : A \to B(H)$ be completely positive (resp. completely J-positive) satisfying statement* (2) (*resp. statement* (3)) *of Theorem* 2.1. *Then $\phi^e(x+\lambda 1) = \phi(x)+\lambda k1$ defines a completely positive (resp. completely J-positive) extension of ϕ.*

Proof. We detail out the case of complete J-positivity, the other one being similar. Let $(\phi^e)_n = \phi^e \otimes id : M_n(A_e) \to B(H) \otimes M_n(\mathbb{C})$. For $x = [x_{ij}] \in M_n(A_e)$, $\phi_n(x^*) = \phi_n([x_{ij}]^*) = \phi_n([x_{ji}^*]) = [\phi(x_{ji}^*)] = [\phi(x_{ji})^*] = [\phi(x_{ij})]^* = \phi_n(x)^*$. Also, $x_{ij} = h_{ij} + \lambda_{ij}$, $h_{ij} \in A$, $\lambda_{ij} \in \mathbb{C}$ so that $x = h + \lambda$ where $h = [h_{ij}] \in M_n(A)$, $\lambda = [\lambda_{ij}] \in M_n(\mathbb{C})$. Suppose $x = x^*$. Then $h_{ij} = h_{ji}^*, \lambda_{ij} = \bar{\lambda}_{ji}$ for all i,j, i.e. $h = h^*, \lambda = \lambda^*$ (hermitian adjoint). Let $\xi = \sum \xi_i \otimes e_i \in H \otimes \mathbb{C}^n, (e_i)$ being the standard orthonormal basis in $\mathbb{C}^n$. It is sufficient to show that $\langle (\phi^e)_n(x^2)\xi, \xi\rangle \geq 0$. Now $x^2 = [\sum_l x_{il}x_{lj}] = [\sum_l (h_{il} + \lambda_{il})(h_{lj} + \lambda_{lj})] = [\sum_l h_{il}h_{lj}] + [\sum_l \lambda_{il}h_{lj}] + [\sum_l h_{il}\lambda_{lj}] + [\sum_l \lambda_{il}\lambda_{lj}] = h^2 + \lambda h + h\lambda + \lambda^2$. Hence $(\phi^e)_n(x^2) = \phi_n(h^2) + \lambda\phi_n(h) + \phi_n(h)\lambda + \lambda^2$ (matrix multiplication). Using Lemma 2.16 (b), we obtain, for each ξ in $H \otimes \mathbb{C}^n$,

$$\begin{aligned}
\langle (\phi^e)_n(x^2)\xi, \xi\rangle &= \langle \phi_n(h^2)\xi, \xi\rangle + \langle \lambda\phi_n(h)\xi, \xi\rangle + \langle \phi_n(h)\lambda\xi, \xi\rangle + k\langle \lambda^2\xi, \xi\rangle \\
&= \langle \phi_n(h^2)\xi, \xi\rangle + \langle \lambda\phi_n(h)\xi, \xi\rangle + \langle \xi, \lambda\phi_n(h)\xi\rangle + k\langle \lambda^2\xi, \xi\rangle \\
&= \langle \phi_n(h^2)\xi, \xi\rangle + \langle \lambda\phi_n(h)\xi, \xi\rangle + \langle \lambda\phi_n(h)\xi, \xi\rangle^- + k\langle \lambda^2\xi, \xi\rangle \\
&\geq \langle \phi_n(h^2)\xi, \xi\rangle - 2|\langle \lambda\phi_n(h)\xi, \xi\rangle| + k\|\lambda\xi\|^2 \\
&\geq \langle \phi_n(h^2)\xi, \xi\rangle - 2\|\phi_n(h)\xi\|\|\lambda\xi\| + k\|\lambda\xi\|^2 \\
&\geq (1/k)\langle \phi_n(h)^2\xi, \xi\rangle - 2\|\phi_n(h)\xi\|\|\xi\| + k\|\lambda\xi\|^2 \\
&= [(1/k^{1/2})\|\phi_n(h)\xi\| - k^{1/2}\|\lambda\xi\|]^2 \geq 0.
\end{aligned}$$

Proof of Theorem 2.1. (1) *implies* (2) and (1) *implies* (3). These are clear.
(2) *implies* (1). Our proof is a CP-analogue of the arguments in [24], applied within the formalism of Stinespring dilation. The vector space $A \otimes H$ is endowed with the non-negative sesquilinear form $\beta(\xi, \eta) = \sum_{i,j}^n \langle \phi(b_j^* a_i)\xi_i, \eta_j\rangle\rangle = \langle \phi_n([b_j^* a_i])(\sum \xi_i \otimes e_i), \sum \eta_i \otimes e_i\rangle$ for $\xi = \sum a_i \otimes \xi_i, \eta = \sum b_j \otimes \eta_j$ in $A \otimes H$. Then $N = \{\xi = \sum a_i \otimes \xi_i \in A \otimes H : \beta(\xi, \xi) = \sum \langle \phi(a_j^* a_i)\xi_i, \xi_j\rangle = 0\}$ is a subspace. Let $J : A \otimes H \to A \otimes H/N$, $J\xi = \xi + N$. Then $\langle J\xi, J\eta\rangle = \beta(\xi, \eta)$ makes $A \otimes H/N$ into an inner product space whose completion is denoted by K. Define an action π_o of A on $A \otimes H$ by $\pi_o(a)(\sum x_i \otimes \xi_i) = \sum ax_i \otimes \xi_i$. The admissibility of a positive functional on a Banach $*$-algebra [21, Theorem. 4.5.2, p. 214] is used to conclude that $\beta(\pi_o(a)\xi, \pi_o(a)\xi) \leq \|a\|^2\beta(\xi, \xi)$. Indeed, taking $\tilde{a} = [\delta_{ij}a], \tilde{x} = [\delta_{1i}x_j]$ in $M_n(A)$,

$$\begin{aligned}\beta[\pi_o(a)\xi,\pi_o(a)\xi] &= \sum_{i,j}\langle\phi(x_j^* a^* a x_i)\xi_i,\xi_j\rangle\\ &= \sum\langle\phi((\tilde{x}^*\tilde{a}^*\tilde{a}\tilde{x})_{j,i})\xi_i,\xi_j\rangle\\ &= \left\langle\phi_n(\tilde{x}^*\tilde{a}^*\tilde{a}\tilde{x})\sum\xi_i\otimes e_i,\sum\xi_i\otimes e_i\right\rangle\\ &\le \|\tilde{a}\|^2\left\langle\phi_n(\tilde{x}^*\tilde{x})\sum\xi_i\otimes e_i,\sum\xi_i\otimes e_i\right\rangle = \|a\|^2\beta(\xi,\xi),\end{aligned}$$

the last inequality being a consequence of the fact that a positive linear functional f on a Banach $*$-algebra with isometric involution satisfies $f(v^*u^*uv) \le \|u\|^2 f(v^*v)$. One can further verify $\pi_o(ab) = \pi_o(a)\pi_o(b), \beta(\pi_o(a)\xi,\eta) = \beta(\xi,\pi_o(a^*)\eta)$. It follows that for any $a \in A, \pi(a)J\xi = J\pi_o(a)\xi$ gives a well defined bounded linear operator $\pi(a)$: $K \to K$, $\|\pi(a)\| \le \|a\|$, and $\pi : A \to B(K)$ is a $*$-homomorphism.

Now consider the linear map $F : A\otimes H/N \to H$, $FJ\xi = \sum\phi(x_i)\xi_i$ for $\xi = \sum x_i\otimes\xi_i$ in $A\otimes H$. Then

$$\begin{aligned}\|FJ\xi\|^2 &= \left\langle\sum\phi(x_i)\xi_i,\sum\phi(x_j)\xi_j\right\rangle = \sum_{i,j}\langle\phi(x_j)^*\phi(x_i)\xi_i,\xi_j\rangle\\ &= \left\langle[\phi(x_j)^*\phi(x_i)]\sum\xi_i\otimes e_i,\sum\xi_i\otimes e_i\right\rangle\\ &= \left\langle\phi_n(x)^*\phi_n(x)\sum\xi_i\otimes e_i,\sum\xi_i\otimes e_i\right\rangle \quad (x = [\delta_{1i}x_j])\\ &\le k\left\langle\phi_n(x^*x)\sum\xi_i\otimes e_i,\sum\xi_i\otimes e_i\right\rangle \quad \text{(Lemma 2.16)}\\ &= k\left\langle[\phi(x_j^*x_i)]\sum\xi_i\otimes e_i,\sum\xi_1\otimes e_i\right\rangle\\ &= k\sum_{i,j}\langle\phi(x_j^*x_i)\xi_i,\xi_j\rangle = k\langle J\xi,J\xi\rangle = k\|J\xi\|^2.\end{aligned}$$

This gives a bounded linear operator $F : K \to H$ satisfying $\|F\eta\| \le k^{1/2}\|\eta\| (\eta \in K)$. Now let $V = F^*$. Then for all $h \in H$, $\xi = \sum x_i\otimes\xi_i \in A\otimes H$, $a \in A$, we have $\langle\sum\phi(x_i)\xi_i,h\rangle = \langle F(J\xi),h\rangle = \langle J\xi,Vh\rangle = \langle\sum x_i\otimes\xi_i,Vh\rangle$ and $\langle\sum\phi(ax_i)\xi_i,h\rangle = \langle\sum x_i\otimes\xi_i,\pi(a^*)Vh\rangle$. Next we claim that $J(b\otimes\eta) = \pi(b)V\eta (b\in A,\eta\in H)$. Indeed, taking ξ as above, $\langle J\xi,J(b\otimes\eta)\rangle = \langle J(\sum x_i\otimes\xi_i),J(b\otimes\eta)\rangle = \sum\langle\phi(b^*x_i)\xi_i,\eta\rangle = \langle\sum\phi(b^*x_i)\xi_i,\eta\rangle = \langle\sum x_i\otimes\xi_i,\pi(b)V\eta\rangle$. Since ξ is arbitrary in $A\otimes H$, and since $J(A\otimes H)$ is dense in K, it follows that $J(b\otimes\eta) = \pi(b)V\eta\,(b\in A,\eta\in H)$. This immediately gives span $(\pi(A)VH) = \text{span } J(A\otimes H)$, $[\pi(A)VH] = K$. Finally, for any $a \in A$, and η and η' in H, we have $\langle V^*\pi(a)V\eta,\eta'\rangle = \langle\pi(a)V\eta,V\eta'\rangle = \langle\pi(a\otimes\eta),V\eta'\rangle = FJ(a\otimes\eta),\eta'\rangle = \langle\phi(a)\eta,\eta'\rangle$ showing that $\phi(a) = V^*\pi(a)V$. It follows that ϕ is Stinespring representable.

(3) *implies* (1). In the notations of the proof of Lemma 2.15, the τ-continuous map $\bar{\phi} : \bar{A} \to B(H)$ is hermitian, positive, and it satisfies statement (3) on $\bar{A}$. Further, $(\bar{\phi})_n$ (denoted by $\bar{\phi}_n$) also satisfies $\bar{\phi}_n(h)^2 \le k\bar{\phi}_n(h^2)$ for all $h = h^*$ in $M_n(A)$. This follows from Lemma 2.16, together with the denseness of $M_n(A)$ – as well as the continuity of $\bar{\phi}_n$ – in the projective cross-norm $\gamma = \tau\otimes\|\ \|$ on $\bar{A}\otimes M_n(\mathbb{C})$. By Lemma 2.15, the map $\bar{\phi}^e : (\bar{A})_e \to B(H)$, $\bar{\phi}^e(x+\lambda 1) = \bar{\phi}(x) + \lambda k1$ is positive, which by Lemma 2.17, is completely J-positive. Now as $\bar{A}$ is symmetric, $M_n(\bar{A})$ is also symmetric. It follows from Lemma 2.13 that $\bar{\phi}^e$ is completely positive on the unital Banach $*$-algebra $\bar{A}_e$. By the result of Evans [9, Theorem 2.13], $\bar{\phi}^e$ is Stinespring representable. Hence by (1) imp. (2)

shown earlier, $\bar{\phi}^e(x)^*\bar{\phi}^e(x) \leq m\bar{\phi}^e(x^*x)(x \in \bar{A}_e)$ for some $m > 0$. Thus $\phi(x)^*\phi(x) \leq m\,\phi(x^*x)(x \in A)$; and by (2) implies (1) shown earlier, ϕ is Stinespring representable.

(1) *implies* (4). This is clear.

(4) *implies* (1). The extension $\phi^e : A_e \to B(H)$ has to be of the form $\phi(x+\lambda 1) = \phi(x) + \lambda\phi^e(1)$. Then ϕ^e is Stinespring representable, so that, in self-explanatory notations, $V^*\pi(x)V = \phi^e(x)(x \in A_e), [\pi(A_e)VH] = K$. Let $K_1 = [\pi(A)VH]$. Since $\pi(A)K_1 \subset K_1$, the projection $P : K \to K_1$ is in the commutant $\pi(A)'$ of $\pi(A)$. Let $\sigma : A \to B(K_1)$ be $\sigma(x) = \pi(x)|_{K_1}$. Then $\pi(x) = P\sigma(x)P = \sigma(x)P$. Let $V_1 = PV$. Then for all $x \in A, \phi(x) = V^*\pi(x)V = V^*P\sigma(x)PV = V_1^*\sigma(x)V_1$. Further, $\sigma(A)V_1H = \sigma(A)PVH = P\sigma(A)PVH = \pi(A)VH$, hence $[\sigma(A)V_1H] = K_1$.

(5) *implies* (6). This is obvious.

(1) *implies* (5). If $\phi(x) = V^*\pi(x)V$, then $\|\phi(x)\| \leq \|V\|^2\|\pi(x)\| \leq \|V\|^2 p_\infty(x)$.

(5) *implies* (1). The statement (5) implies that ϕ factors through $C^*(A)$ giving completely positive map $\tilde{\phi} : C^*(A) \to B(H)$ satisfying $\tilde{\phi} \circ j = \phi$. The conclusion follows easily.

The remaining assertions follow from Lemma 2.8 (d) and Lemma 2.9.

(2.19) For the proof of Theorem 2.2, note that (2) implies (1) follows from Lemma 2.15. That (3) iff (4) is obvious; whereas (4) implies (2) follows from Kadison's inequality for $\tilde{\phi}$. That (1) implies (4) follows as in first paragraph in the proof of Lemma 2.16.

(2.20) For the proof of Corollary 2.3, note that (a) follows from Lemma 2.8 and Theorem 2.2. For (b), Lemma 2.6 implies that $\phi(x)^*\phi(x) \leq (\lim \phi(e_i^* e_i))\phi(x^*x) = \|\phi\|\phi(x^*x)(x \in A)$; and Theorem 2.1 applies.

Theorem 2.21. (*The following is a slightly modified version of* 2.1.) *Let A be a complex $*$-algebra. Let $\phi : A \to B(H)$ be a completely positive map. The following are equivalent.*

(a) *ϕ is Stinespring representable.*

(b) (i) *There exists a scalar $k > 0$ such that $\phi(x^*)\phi(x) \leq k\phi(x^*x)(x \in A)$.*

(ii) *There exists a scalar $m > 0$ and a submultiplicative seminorm p on A such that $\|\phi(x)\| \leq mp(x)(x \in A)$.*

3. Applications and related results

(I) *Cauchy–Schwarz inequalities*: A positive functional f on A satisfies $|f(y^*x|^2 \leq f(y^*y)f(x^*x)$ (x, y in A). Let $\phi \in P(A,H)$. If A is unital (or having a BAI), Kadison's inequality $\phi(h)^2 \leq \|\phi\|\phi(h^2)(h = h^* \in A)$ is an operator valued version. If $\phi \in CP(A,H)$, then Lemma 2.6 provides a CP-version, which, in the presence of BAI, reduces to Corollary 2.8 (c). The following contain some other versions of this inequality.

COROLLARY 3.1

(a) *Let $\phi \in CP(A,H)$. Let x, y, z be in A and $t > 0$ be a scalar.*

(1) *Let ϕ be Stinespring representable. Then the following hold.*

(i) $\phi(y^*x)^*(t + \phi(y^*y))^{-1}\phi(y^*x) \leq \phi(x^*x)$.

(ii) $\phi(x^*)\phi(x) \leq \|\phi^e(1)\|p_\infty(x)^2\phi^e(1)$.

(iii) $\phi(x^*x) \leq \|\phi^e(1)\|p_\infty(x)^2 1$.

(2) *The following hold.*

(i) $\phi(y^*xy)^*\phi(y^*xy) \leq p_\infty(x)^2\|\phi(y^*y)\|\phi(y^*y)$.

(ii) $\phi(y^*xy)^*\phi(y^*xy) \leq \|\phi(y^*y)\|\phi(y^*x^*xy) \leq \|\phi(y^*y)\|p_\infty(x)^2\phi(y^*y)$.

(iii) $\phi(z^*y)^*xz^*[t+\phi(z^*y^*yz)]^{-1}\phi(z^*y^*xz) \leq \phi(z^*x^*xz)$.

(b) *If* $\phi \in P(A,H)$, *then* $\phi(y^*hy)^2 \leq \|\phi(y^*y)\|\phi(y^*h^2y)(y\in A, h=h^*$ *in* $A)$; *and* $\phi(y^*x^*xy) \leq p_\infty(x)\phi(y^*y)$. *Further, if* $\phi \in CP(A,H)$, *then* $\phi(y^*xy)^*\phi(y^*xy) \leq \|\phi(y^*y)\|s(x)^2\phi(y^*y)$. *where* $s(x) = r(x^*x)^{1/2}$, $r()$ *being the spectral radius.*

(c) *If* $\phi \in P(A,H)$ *and has abelian range, then*

$$\phi(y^*x)^*\phi(y^*x) \leq \phi(y^*y)\phi(x^*x) \quad \textit{for all } x,y \textit{ in } A.$$

Does above (1) (i) hold if ϕ is not Stinespring representable ? Above (b) is an operator valued version of the familiar [4, Lemma 37.6 (iii)].

Proof. (a) (1). Inequality (i) can be proved as in [9, Theorem 1.14, p. 15] for positive definite kernels. Representability of $f_\xi, f_\xi(x) = \langle\phi(x)\xi,\xi\rangle$, Theorem 2.1 and Lemma 2.11 (b) (ii) give $\langle\phi(x)^*\phi(x)\xi,\xi\rangle \leq \|\phi(1)\|\langle\phi\langle x^*x\rangle\xi,\xi\rangle \leq \|\phi(1)\|p_\infty(x)^2\langle\phi(1)\xi,\xi\rangle$ which gives (ii). This in turn implies $\|\phi(x)\| \leq \|\phi(1)\|p_\infty(x)$, hence $\phi(x^*x) \leq \|\phi(x^*x)\|1 \leq \|\phi^e\|p_\infty(x)^2 1$. (2) For $y \in A$, let $\phi_y(x) = \phi(y^*xy)$. Then ϕ_y is completely positive, since $(\phi_y)_n = (\phi_n)Y$ where $Y = [\delta_{ij}y]$. Thus $\phi_y(x+\lambda 1) = \phi_y(x)+\lambda\phi(y^*y)$ gives a completely positive extension to A_e. Thus ϕ_y is Stinespring representable. Hence (2) follows from (1). Also (b) follows as above from Lemma 2.11 (b) (i). (c) can be easily proved.

One may look for analogue of above inequalities for J-positive and completely J-positive maps.

A dilation-free approach to the CP-Schwarz inequality

Let A be a C^*-algebra. Let $\phi \in CP(A,H)$. The only proof of the CP-Schwarz inequality $\phi(x)^*\phi(x) \leq \|\phi\|\phi(x^*x)$ $(x \in A)$ that the author knows is as in [27, ch. IV, Corollary 3.8, p. 199], which is based on Stinespring dilation. The following provides a dilation-free proof in a more general context. It uses Kadison's Schwarz inequality in a crucial way via Corollary 2.3; and exhibits the essential difference between these two inequalities.

COROLLARY 3.2

(a) *Let* A *be a Banach* $*$*-algebra with BAI. Let* $\phi : A \to B(H)$ *be a 2-positive map. Then* $\phi(x)^*\phi(x) \leq \|\phi\|\phi(x^*x)$ *for all* x *in* A.

(b) *There exists a* C^**-algebra* A *and a positive, non-2-positive map* $\phi : A \to B(H)$ *for an appropriate* H *such that for no scalar* $k > 0$, $\phi(x)^*\phi(x) \leq k\phi(x^*x)$ *hold for all* x *in* A.

Proof. (a) The algebra $M_2(A)$ has BAI. By Corollary 2.3, ϕ is extendable, hence hermitian; and the positive map $\phi_2 : M_2(A) \to M_2(B(H) \subset B(H\otimes\mathbb{C}^2)$, $\phi_2 = \phi\otimes id$, satisfies $\phi_2(h)^2 \leq \|\phi\|\phi_2(h^2)$ for all $h = h^*$ in $M_2(A)$. Let $x \in A, \xi \in H$. Take $h = \begin{bmatrix} 0 & x^* \\ x & 0 \end{bmatrix} = h^*$. Then

$$\begin{bmatrix} \phi(x)^*\phi(x) & 0 \\ 0 & \phi(x)\phi(x)^* \end{bmatrix} \leq \|\phi\| \begin{bmatrix} \phi(x^*x) & 0 \\ 0 & \phi(xx^*) \end{bmatrix}.$$

Taking $\xi = \xi\otimes e_1 + 0\otimes e_2 \in H\otimes\mathbb{C}^2$ ($\{e_1,e_2\}$ =standard basis in $\mathbb{C}^2$),

$$\left\langle \begin{bmatrix} \phi(x)^*\phi(x) & 0 \\ 0 & \phi(x)\phi(x)^* \end{bmatrix} \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \begin{bmatrix} \xi \\ 0 \end{bmatrix} \right\rangle$$
$$\leq \|\phi\| \left\langle \begin{bmatrix} \phi(x^*x) & 0 \\ 0 & \phi(xx^*) \end{bmatrix} \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \begin{bmatrix} \xi \\ 0 \end{bmatrix} \right\rangle.$$

Hence $\langle \phi(x)^*\phi(x)\xi, \xi\rangle \leq \|\phi\|\langle \phi(x^*x)\xi, \xi\rangle$.
(b) Let $A = M_2(\mathbb{C}), \phi : M_2(\mathbb{C}) \to M_2(\mathbb{C}) = B(\mathbb{C}^2)$ be $\phi(x) = \mathrm{tr}(x)1 - x$. Then ϕ is known to be positive, but not 2-positive. Let $k > 0$. Suppose $\phi(x)^*\phi(x) \leq k\phi(x^*x)(x \in A)$. Then $\mathrm{tr}(x)^*\mathrm{tr}(x)1 - \mathrm{tr}(x)^*x - \mathrm{tr}(x)x^* + x^*x \leq k(\mathrm{tr}(x^*x)1 - x^*x)$. Taking $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathrm{tr}(x) = \mathrm{tr}(x^*) = 0, x^*x = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \mathrm{tr}(x^*x) = 1$. Thus $(k+1)x^*x \leq k1$. Hence for all $z \in \mathbb{C}$,

$$\left\langle \begin{bmatrix} 0 & 0 \\ 0 & k+1 \end{bmatrix} \begin{bmatrix} 0 \\ z \end{bmatrix}, \begin{bmatrix} 0 \\ z \end{bmatrix} \right\rangle \leq \left\langle \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} 0 \\ z \end{bmatrix}, \begin{bmatrix} 0 \\ z \end{bmatrix} \right\rangle,$$

giving $(k+1)|z|^2 \leq k|z|^2$, a contradiction.

(II) *Automatic Stinespring representability*: There are three closely related aspects of positive functionals viz. admissibility [21 p. 213], continuity and representability. Admissibility is automatic in Banach $*$-algebras [4, Lemma 37.6, p. 197; 21. Theorem. 4.5.2] (but not in topological $*$-algebras); automatic continuity has been considerably discussed in the literature encompassing more general topological $*$-algebra case; whereas automatic representability seems to have received least attention even in Banach $*$-algebras. Note that representability is stronger than continuity.

COROLLARY 3.3

Assume that $A = A^2$, *i.e.* $A = \mathrm{span}\,(yx : x, y \text{ in } A)$.

(a) *Assume the following:*

(a1) *Every non-zero member of* $CP(A,H)$ *dominates a non-zero Stinespring representable member of* $CP(A,H)$.
(a2) *Given* $\phi \in CP(A,H)$ *and letting* $S(\phi) = (\psi \in CP(A,H) : \psi$ *is Stinespring representable and* $\phi \geq \psi)$, *there exists a scalar* $k = k_\phi > 0$ *such that* $\psi(x)^*\psi(x) \leq k\psi(x^*x)$ *holds for all* x *and for all* $\psi \in S(\phi)$.

Then every ϕ *in* $CP(A,H)$ *is Stinespring representable.*

(b) *Assume the following:*

(b1) *Every non-zero member of* $P(A,H)$ *dominates a non-zero extendable member of* $P(A,H)$.
(b2) *Given* $\phi \in P(A,H)$ *and letting* $P(\phi) = (\psi \in P(A,H) : \psi$ *is extendable and* $\phi \geq \psi)$, *there exists* $k > 0$ *such that* $\psi(h)^2 \leq k\psi(h^2)$ *for all* $h = h^*$, *all* ψ *in* $P(\phi)$.

Then every ϕ *in* $P(A,H)$ *is extendable.*

(c) *Assume the following:*

(c1) *Every non-zero positive functional on* A *dominates a non-zero representable positive functional.*
(c2) *Above (b2) holds with* $H = \mathbb{C}$.

Then every positive functional on A is representable.

Even in the scalar case, above (c) gives representability analogue of the automatic continuity theorem [4, Theorem 37.13]. In the following (V), we discuss (i) examples showing that in above, assumptions (a2) (and similarily (b2) and (c2) can not be omitted; and (ii) examples in which every positive functional is continuous, but not each such functional is representable.

Lemma 3.4. *Let ϕ_1 and ϕ_2 in $CP(A,H)$ be Stinespring representable. Then $\phi_1+\phi_2$ is also Stinespring representable.*

Proof. Clearly $\phi_1+\phi_2\in CP(A,H)$. By Theorem 2.1, there exists $k_1>0, k_2>0$ such that for all $x\in A$, $\phi_1(x)^*\phi_1(x)\le k_1\phi_1(x^*x)$, $\phi_2(x)^*\phi_2(x)\le k_2\phi_2(x^*x)$. Let $\phi=\phi_1+\phi_2$. Then

$$\begin{aligned}\phi(x)^*\phi(x) &= \{(\phi_1(x)^*+\phi_2(x)^*\}\{(\phi_1(x)+\phi_2(x))\}\\ &= \phi_1(x)^*\phi_1(x)+\phi_1(x)^*\phi_2(x)+\phi_2(x)^*\phi_1(x)+\phi_2(x)^*\phi_2(x).\end{aligned}$$

Hence, for any $\xi\in H$,

$$\begin{aligned}\langle\phi(x)^*\phi(x)\xi,\xi\rangle &= \|\phi_1(x)\xi\|^2+\|\phi_2(x)\xi\|^2+\langle\phi_2(x)\xi,\phi_1(x)\xi\rangle+\langle\phi_1(x),\phi_2(x)\xi\rangle\\ &\le \|\phi_1(x)\xi\|^2+\|\phi_2(x)\xi\|^2+2|\langle\phi_1(x)\xi,\phi_2(x)\xi\rangle|\\ &\le \|\phi_1(x)\xi\|^2+\|\phi_2(x)\xi\|^2+2\|\phi_1(x)\xi\|\,\|\phi_2(x)\xi\|\\ &= (\|\phi_1(x)\xi\|+\|\phi_2(x)\xi\|)^2\\ &= \{\langle\phi_1(x)^*\phi_1(x)\xi,\xi\rangle^{1/2}+\langle\phi_2(x)^*\phi_2(x)\xi,\xi\rangle^{1/2}\}^2\\ &\le \{k_1^{1/2}\langle\phi_1(x^*x)\xi,\xi\rangle^{1/2}+k_2^{1/2}\langle\phi_2(x^*x)\xi,\xi\rangle^{1/2}\}^2\\ &\le \max(k_1,k_2)\{\langle\phi_1(x^*x)\xi,\xi\rangle^{1/2}+\langle\phi_2(x^*x)\xi,\xi\rangle^{1/2}\}^2\\ &\le \max(k_1,k_2)\{2\langle\phi(x^*x)\xi,\xi\rangle^{1/2}\}^2\le 4\max(k_1,k_2)\langle\phi(x^*x)\xi,\xi\rangle.\end{aligned}$$

The conclusion now follows from Theorem 2.1.

Proof of Corollary 3.3. We give details for part (a). Since $A^2=A$, each $a\in A$ is of form $a=\sum u_jv_j$ with u_j,v_j in A. Also, for any u,v in A, $4uv=(v+u^*)^*\,(v+u^*)-(v-u^*)^*\,(v-u^*)+i(v+iu^*)^*(v+iu^*)-i(v-iu^*)^*(v-iu^*)$. It follows that

$$A=\operatorname{span}A^+. \tag{1}$$

Now consider the order relation $\psi\le\phi$ in $CP(A,H)$, where $\psi\le\phi$ means $\phi-\psi\in CP(A,H)$. Clearly, this is reflexive and transitive. Since $A=\operatorname{span}A^+$, it follows that $\le$ is antisymmetric also, hence is a partial order. Let $\phi\in CP(A,H)$. Let $S(\phi)=\{\psi\in CP(A,H):\psi$ is Stinespring representable and $\psi\le\phi\}$. We show that the partially ordered set $(S(\phi),\le)$ has maximal element.

Let C be any chain in $(S(\phi),\le)$. Let $a\in A$, say $a=\sum\propto_j b_j^*b_j$, finite sum, $\propto_j$ scalars. Then for all $b\in A$, $\psi\in C$, $\xi\in H$, we have

$$\langle\psi(b^*b)\xi,\xi\rangle\le\langle\phi(b^*b)\xi,\xi\rangle. \tag{2}$$

We first show that

$$\phi'(a)=\lim_{\psi\in C}\psi(a)\quad(a\in A)\quad\text{is defined.} \tag{3}$$

By (2), $\lim_{\psi\in C}\langle\psi(b^*b)\xi,\xi\rangle$ exists in $\mathbb{R}$ for all $b\in A,\xi\in H$. Then

$$\begin{aligned}\lim_{\psi\in C}\langle\psi(a)\xi,\xi\rangle &= \lim_{\psi\in C}\Big\langle\psi\Big(\sum\propto_j b_j^*b_j\Big)\xi,\xi\Big\rangle = \lim_{\psi\in C}\sum\propto_j\langle\psi(b_j^*b_j)\xi,\xi\rangle\\ &\le\sum\propto_j\langle\phi(b_j^*b_j)\xi,\xi\rangle = \langle\phi(a)\xi,\xi\rangle.\end{aligned} \tag{4}$$

This defines $\omega_\xi(a)=\lim_{\psi\in C}\langle\psi(a)\xi,\xi\rangle$. Clearly ω_ξ is positive linear functional on A. Now, the polarization identity, for any $T\in B(H)$,

$$\begin{aligned}4\langle T\xi,\eta\rangle =&\langle T(\xi+\eta),\xi+\eta\rangle-\langle T(\xi-\eta),\xi-\eta\rangle+i\langle T(\xi+i\eta),\xi+i\eta\rangle\\ &-i\langle T(\xi-i\eta),\xi-i\eta\rangle\end{aligned}$$

gives, using (4), that for any ξ,η in $H,a\in A$,

$$B_a(\xi,\eta)=\lim_{\psi\in C}\langle\psi(a)\xi,\eta\rangle \text{ exists.} \tag{5}$$

It is easily seen that $(\xi,\eta)\to B_a(\xi,\eta)$ defines a sesquilinear form on H. Further, it is a bounded sesquilinear form. Indeed, for any $\psi\in C,a=\sum\propto_j b_j^*b_j$ in A,

$$\|\psi(a)\|\le\sum\propto_j\|\psi(b_j^*b_j)\|\le\sum\propto_j\|\phi(b_j^*b_j)\|=M(a,\phi)=M \text{ (say)}.$$

Hence, by the uniform boundedness principle, there is $k>0$ such that $\|\psi\|\le k$ for all $\psi\in C$. Hence, in (5), we get

$$|B_a(\xi,\eta)|\le k\|a\|\|\xi\|\eta\|. \tag{6}$$

showing that $B_a(.,.)$ is a bounded sesquilinear form on H. This defines $\phi'(a)\in B(H)$ such that $\langle\phi'(a)\xi,\eta\rangle=B_a(\xi,\eta)$, ($\xi,\eta$ in H). This proves (3).

The mapping $\phi':A\to B(H)$ defined above is linear. Since each $\psi\in C$ is Stinespring representable, hence hermitian, it follows that $\phi'(a^*)=\phi'(a)^*(a\in A)$. Further, (6) implies that ϕ' is continuous. Lemma 2.4 (e) implies that $\phi'\in CP(A,H)$ satisfying $\phi'\le\phi$. Further, for all $a\in A,\xi$ in H,

$$\begin{aligned}\langle\phi'(x)^*\phi'(x)\xi,\xi\rangle &= \|\phi'(x)\xi\|^2=\lim_{\psi\in C}\|\psi(x)\xi\|^2=\lim_{\psi\in C}\langle\psi(x)^*\psi(x)\xi,\xi\rangle\\ &\le k_\phi\lim_{\psi\in C}\langle\psi(x^*x)\xi,\xi\rangle\le k_\phi\langle\phi'(x^*x)\xi,\xi\rangle.\end{aligned}$$

Hence ϕ' is Stinespring representable by Theorem 2.1. Thus $\phi'\in C$ and is an upper bound for C.

By Zorn's lemma, $S(\phi)$ admits a maximal element, say ψ_o. We show that $\phi=\psi_0$. If $\phi-\psi_o\ne 0$, then by assumption (a1), there exists a Stinespring representable $\psi_1\in CP(A,H)$ such that $\phi-\psi_o\ge\psi_1$. Thus $\phi\ge\psi_o+\psi_1$; and $\psi_o+\psi_1$ is Stinespring representable completely positive map by Lemma 3.4. Thus $\psi=\psi_o+\psi_1\in S(\phi),\psi\ne\psi_o$, contradicting the maximality of ψ_o. Hence $\phi=\psi_o$ is Stinespring representable.

(III) *Extension problem:* By a celebrated extension theorem due to Arveson [1, Theorem 1.2.3], if B is a closed self-adjoint subspace of a unital C^*-algebra A, and if $1\in B$, then any $\phi\in CP(B,H)$ extends to a $\phi\in CP(A,H)$. The following has a bearing with this.

COROLLARY 2.6

Let A be a Banach $*$-algebra with BAI. Let B be a $*$-subalgebra of A. Assume that at least one of the following holds.

(a) *A is hermitian (in particular, A is spectrally invariant in $C^*(A)$) and B is closed in A.*
(b) *B is a Banach $*$-algebra with some norm such that $C^*(B) \rightarrow C^*(A)$ injectively. In particular, the Banach $*$-algebra B is dense and spectrally invariant in A so that $C^*(B) = C^*(A)$.*

Let $\phi \in CP(B,H)$. Then ϕ extends to a $\tilde{\phi} \in CP(A,H)$ iff ϕ is Stinespring representable. If A is unital and B contains identity of A, then under above assumption, any $\phi \in CP(B,H)$ extends to a $\phi \in CP(A,H)$.

Proof. Let $\phi \in CP(B,H)$. Assume (a). Let $\tilde{\phi} \in CP(A,H)$ be such that $\tilde{\phi}|_B = \phi$. By Corollary 2.3, $\tilde{\phi}$ is Stinespring representable; hence by Theorem 2.1, $\phi(x)^*\phi(x) \leq k\phi(x^*x)(x \in B)$. Conversely, let ϕ be Stinespring representable, hence extends to a $\phi^e \in CP(B_e,H)$; and there exists a Hilbert space K, a $*$-representation $\pi : B_e \rightarrow B(K)$ and a bounded linear $V : H \rightarrow K$ such that $\phi^e(x) = V^*\pi(x)V(x \in B_e)$. Now A_e is hermitian [21, Theorem 4.7.9, p. 223]; and B_e is a $*$-subalgebra of A_e containing the identity. By Lemma [21, Theorem 4.7.20], there exists a Hilbert space W containing K as a closed subspace and a $*$-representation $\sigma : A_e \rightarrow B(W)$ such that $\sigma(x)|_K = \pi(x)(x \in B_e)$. Let $id : K \rightarrow W$ be the inclusion. Then $P = (id)^* : W \rightarrow K$ is the orthogonal projection. Let $V' = id \circ V : H \rightarrow W$. Define $\phi'(x) = (V')^*\sigma(x)V'(x \in A_e)$. Then $\phi' \in CP(A_e,H)$ and for all $x \in B$, $\phi'(x) = V'^*\sigma(x)V' = V'^*P\sigma(x)id\, V' = V'^*\pi(x)V' = \phi(x)$.

Now assume (b). One way conclusion is obvious as in the above case. Let $\phi \in CP(B,H)$ be Stinespring representable, hence extends to $\phi^e \in CP(B_e,H)$. By Lemmas 2.7, 2.8, ϕ^e factors through $C^*(B_e)$ as $\phi^e = \tilde{\phi}^e \circ j_B$ with $\tilde{\phi}^e \in CP(C^*(B_e),H)$. By Lemma 2.10, $C^*(B_e) \rightarrow C^*(A_e)$ injectively, and $C^*(B_e)$ is a C^*-subalgebra of $C^*(A_e)$ containing the identity of $C^*(A_e)$. By Arveson extension Theorem, [1, Theorem 1.2.3] $\tilde{\phi}^e$ extends as a $\psi \in CP(C^*(A_e),H)$. Then $\lambda = \psi \circ j_{A_e}|A \in CP(A,H)$ and λ is an extension of ϕ. This completes the proof.

Can the additional assumptions (a) or (b) in above Corollary 2.6 be weakened? Corollary 2.6 (a) applies to closed $*$-subalgebras B of the group algebra $A = L^1(G)$ of a locally compact group G which is symmetric (i.e., $L^1(G)$ is symmetric). A Segal $*$-algebra in a Banach $*$-algebra $(A, \|\cdot\|)$ is a dense $*$-ideal B of A that is a Banach $*$-algebra with some norm $|\cdot|$ such that the inclusion $(B, |\cdot|) \rightarrow (A, \|\cdot\|)$ is continuous. By [8, Theorem 31.5], B can not have BAI.

COROLLARY 3.7

Let A have BAI. Let B be a Segal $*$-algebra in A. Let $\phi \in CP(B,H)$. The following are equivalent.

(1) *ϕ extends to a $\tilde{\phi} \in CP(A,H)$.*
(2) *ϕ is Stinespring representable.*
(3) *ϕ is continuous in the norm of A.*

Further, if $A = L^1(G)$ for a locally compact group G, then each of above is equivalent to

(4) *There exists a strongly continuous completely positive definite function $x : g \in G \rightarrow x(g) \in B(H)$ such that $\phi(f) = \int_G f(g)x(g)\mathrm{d}g(f \in B)$.*

Let G be a locally compact group. Let $x : G \rightarrow B(H)$ be a weakly continuous function. Recall that x is positive definite (resp. completely positive definite) if for every $n \in \mathbb{N}$, each $s_1, \ldots, s_n$ in G and each scalars $\lambda_1, \ldots, \lambda_n$ (resp. each $T_1, \ldots, T_n$ in $B(H)$), it holds that $\sum_{ij} \lambda_i\bar{\lambda}_j x(s_j^{-1}s_i) \geq 0$ (resp. $\sum_{ij} T_j^* x(s_j^{-1}s_i)T_i \geq 0$) in $B(H)$.

Proof. Since B is an ideal in $A, B_{-1} = A_{-1} \cap B$, where K_{-1} denote the set of all quasi-regular elements of K. It follows that B is a Q-normed algebra in the norm $\|\cdot\|_A$ of A, i.e. B_{-1} is open in $\|\cdot\|_A$. By [12, Theorem 3.1], each $*$-representation π on B is $\|\cdot\|_A$ continuous; and, by the density of B in A, π extends uniquely to a $*$-representation of A. It follows that A_e and B_e have the same collection of $*$-representations, identified via restriction. Thus srad $B_e =$ (srad $A_e \cap B_e$) [4, Theorem 40.9, p. 223], and $p_{\infty|B_e}^{A_e} = p_\infty^{B_e}$ (for Gelfand–Naimark pseudonorms). It follows that $C^*(B_e)$ is canonically embedded as a $*$-subalgebra of $C^*(A_e)$. Then $(2) \Leftrightarrow (1)$ by Corollary 3.5. That $(1) \Leftrightarrow (3)$ is due to continuity, as A has BAI. That (3) iff (4) follows from Stinespring representability on $L^1(G)$ (as it has BAI), the correspondence between unitary representations of G and $*$-representation of $L^1(G)$ and Naimark–Sz. Nagy characterization of completely positive definite functions [9, Corollary 2.6, p. 17], [27, Ex. 2, p. 203]. This completes the proof.

Let A be a $*$-subalgebra of $L^1(G)$. Suppose either G is symmetric and A is closed; or A is a Banach $*$-algebra, dense and spectrally invariant in $L^1(G)$. It follows from Corollaries 3.6 and 3.7 that every completely positive map on A (in particular, on $L^1(G)$) can be extended to a completely positive map on the measure algebra $M(G)$ and is given by a completely positive definite function on G.

(IV) *Integral representations and operator valued Bochner Theorem*: The classical Bochner Theorem states that if ϕ is a positive definite function on $\mathbb{R}^n$, then there exists a positive Radon measure μ on $\mathbb{R}^n$ having mass $\mu(\mathbb{R}^n) = \phi(0)$ such that $\phi(g) = \int_{\mathbb{R}^n} \exp\{i\langle g, \xi\rangle\} \mathrm{d}\mu(\xi) (g \in \mathbb{R}^n)$. More generally, a positive definite function ϕ on a locally compact abelian group G is determined as a positive Randon measure on the dual group $\hat{G}$ by the formula $\phi(g) = \int_{\hat{G}} x(g) \mathrm{d}\mu(x)$. Via $L^1(G)$, this determines positive linear functionals on the Banach $*$-algebra $L^1(G)$; and becomes a special case of the abstract Bochner–Weil–Raikov integral representation [11, ch. IV, Theorem 21.2; 15, Theorem 33.2] stating that a continuous linear functional f on a commutative Banach $*$-algebra A is positive and representable iff there exists a positive Borel measure μ on the hermitian Gelfand space $\mathfrak{M}^*(A) = \{\varphi \in A' : \varphi \text{ is multiplicative}, \varphi(x^*) = \varphi(x)^- \text{ for all } x\}$ such that $f(x) = \int_{\mathfrak{M}^*(A)} \hat{x}(\varphi) \mathrm{d}\mu(\varphi) (x \in A), \hat{x}$ denoting the Gelfand transform. The following is an operator valued version of this. It also provides a commutative Banach $*$-algebra analogue of the Naimark dilation theorem [26, Theorem 7.5, p. 153] which is a forerunner of Stinespring theorem. Recall that a semispectral measure on a topological space X is a mapping F from Borel σ-algebra $\mathcal{B}(X)$ into $B(H), \omega \in \mathcal{B}(X) \to F(\omega) \in B(H)$ such that for each ξ in $H, \omega \to \langle F(\omega)\xi, \xi\rangle$ is a bounded positive Borel measure.

COROLLARY 3.8

Let A be a commutative Banach $*$-algebra. Let $\phi : A \to B(H)$ be a linear map.

(A) *Let ϕ be positive. The following are equivalent.*

(1) *ϕ is hermitian and for some scalar $k > 0$, $\phi(h)^2 \leq k\,\phi(h^2)$ for all $h = h^*$ in A.*
(2) *There exists a scalar $k > 0$ such that $\phi(x)^*\phi(x) \leq k\,\phi(x^*x)$ for all $x \in A$.*
(3) *There exists a semispectral measure F on $\mathfrak{M}^*(A)$ such that $\phi(x) = \int_{\mathfrak{M}^*(A)} \hat{x}(f)\, \mathrm{d}F(f)$, $\hat{x}$ denoting the Gelfand transform of x.*
(4) *ϕ is completely positive and Stinespring representable.*

(B) *Let A have BAI. Then* (3) *above is equivalent to*
(5) ϕ is positive.

On the one hand, the following is an operator valued analogue of Bochner's theorem; on the other hand, it is a positive linear map analogue of Stone–Naimark–Ambrose–Godement theorem [20, ch. XV, Theorem 3, p. 489] occupying its proper place midway between the two.

COROLLARY 3.9

Let G be a locally compact abelian group. Let $x : G \to B(H)$ be a weakly continuous function. The following are equivalent.

(1) *x is positive definite.*
(2) *x is completely positive definite.*
(3) *There exists a semispectral measure F on Borel subsets of the dual group $\hat{G}$ such that $x(s) = \int_{\hat{G}} \overline{f(s)} \mathrm{d}F(f)$.*

Proof of Corollary 3.8. (A) That (1) iff (2) iff (4) follow from the results in § 2. Now assume (4). Take extension $\phi^e \in CP(A_e, H)$. Let $\tilde{\phi}^e \in CP(C^*(A_e), H)$ such that $\tilde{\phi}^e \circ j_e = \phi^e$, $j_e(z) = z +$ srad A_e. Then $\tilde{\phi} = \tilde{\phi}^e|_{C^*(A)} \in CP(C^*(A), H)$, $\tilde{\phi} \circ j = \phi$. By Gelfand theory, $C^*(A) = C_o(\mathfrak{M}^*(A))$. $C^*(A_e) = C(X)$, $X = \mathfrak{M}^*(A) \cup \{\infty\}$ being one point compactification of $\mathfrak{M}^*(A)$. By Naimark dilation theorem [26, Theorem 7.5, p. 153], there exists a semispectral measure G on Borel sets of X such that $\tilde{\phi}^e(f) = \int_X f(t) \mathrm{d}G(t)$, $f \in C(X)$. By restriction, G defines a semispectral measure F on $\mathfrak{M}^*(A)$ such that, for all $x \in A$, $\phi(x) = \tilde{\phi}(j(x)) = \int_{\mathfrak{M}^*(A)} f(t) \mathrm{d}F(t)$. Thus (4) $\Leftrightarrow$ (3). If (3) holds, then the positive $\tilde{\phi}(f) = \int_{\mathfrak{M}^*(A)} f(t) \mathrm{d}f(t)$ is completely positive and Stinespring representable, and (2) follows.

Proof of Corollary 3.9. Assume (1). Then $\phi : L^1(G) \to B(H)$, $\phi(f) = \int_G f(s)x(s)\mathrm{d}s$ defines a positive linear map on $L^1(G)$ by $\langle \phi(f)\xi, \eta \rangle = \int_G f(s) \langle x(s)\xi, \eta \rangle \mathrm{d}s (\xi, \eta$ in $H)$. As G is abelian, the Banach $*$-algebra $L^1(G)$ is semisimple and symmetric. Hence $\mathfrak{M}^*(L^1(G)) = \mathfrak{M}(L^1(G)) = \tilde{G}$ (dual group) by usual identification. Also, $L^1(G)$ has BAI. It follows from Corollary 3.7 that for some semispectral measure F on $\hat{G}$, $\phi(f) = \int_{\hat{G}} \hat{f}(t) \mathrm{d}F(t)$. Then, for ξ, η in H, for all $f \in L^1(G)$,

$$
\begin{aligned}
\int_G f(s)\langle x(s)\xi, \eta\rangle \mathrm{d}s &= \langle \phi(f)\xi, \eta \rangle \\
&= \int_G \hat{f}(x) \mathrm{d}\langle F(t)\xi, \eta\rangle \\
&= \int_{\hat{G}} \left(\int_G f(s)\overline{t(s)}\mathrm{d}s \right) \mathrm{d}\langle F(t)\xi, \eta \rangle \\
&= \int_G f(s) \left(\int_{\hat{G}} \overline{t(s)} \mathrm{d}\langle F(t)\xi, \eta\rangle \right) \mathrm{d}s.
\end{aligned}
$$

It follows that $x(s) = \int_{\hat{G}} \overline{t(s)} \mathrm{d}F(t)$ and (3) holds. Now assume (3) viz. $x(s) = \int_{\hat{G}} \overline{t(s)} \mathrm{d}F(t)$ for some semispectral measure F on $\hat{G}$. Then $\phi(f) = \int_{\hat{G}} \hat{f}(t) \mathrm{d}F(t)$ defines a positive linear map on $L^1(G)$. By Corollary 3.7, ϕ is completely positive and Stinespring representable. Let $\{\pi, K, V\}$ be a Stinespring representation of ϕ. By reverting the steps in previous proof, $\phi(f) = \int_G f(s)x(s)\mathrm{d}s (f \in L^1(g))$. Further, there exists a weakly continuous unitary representation $s \to U(s)$ of G on K such that $\pi(f) = \int_G f(s)U(s)\mathrm{d}s$ $(f \in L^1(G))$. Then, for all such f, and ξ, η in K,

$$\int_G f(s)\langle x(s)\xi,\eta\rangle \mathrm{d}s = \langle \phi(f)\xi,\eta\rangle = \langle \pi(f)V\xi, V\eta\rangle$$
$$= \int_G f(s)\langle U(s)V\xi, V\eta\rangle \mathrm{d}s$$
$$= \int_G f(s)\langle V^*U(s)V\xi,\eta\rangle \mathrm{d}s.$$

Hence $\langle x(s)\xi,\eta\rangle = \langle V^*U(s)V\xi,\eta\rangle$. Thus x is completely positive definite; and (2) follows.

(V) *Examples and Remarks*: (3.10) Consider the sequence space $l^p = l^p(\mathbb{N}), 1 \leq p < \infty$. It is a non-unital commutative Banach $*$-algebra with pointwise multiplication, complex conjugation and the norm $\|x\| = (\sum |x_n|^p)^{1/p}$.

(i) $l^p \cdot l^p = \{xy | x, y \text{ in } l^p\}$ is a proper dense subset of l^p [8, p. 113]. Thus l^p is not factorizable, hence it fails to admit BAI. However, $(u_n), u_n = (1, 1, \ldots, 1_n, 0, 0, \ldots)$, constitute unbounded approximate identity for l^p.

(ii) Every positive linear functional on l^p is continuous [23, ch. V, Theorem 5.5, p. 228], and is of the form $f = f_a$ for some $a = (a_n) \in l^q$, $1/p + 1/q = 1$, with $a_n \geq 0$ for all n. where $f_a(x) = \sum a_n x_n$. Not every such f is representable. In fact, f_a is representable iff $a \in l^1$. Indeed, $|f_a(x)|^2 \leq k f_a(x^*x)(x \in l^p)$ gives, taking $x = u_n$, that $(a_n) \in l^1$.

(iii) Above (i) and (ii) illustrate that the boundedness of approximate identity can not be omitted from Corollary 2.3.

(iv) By [6], every positive functional on a separable commutative Banach $*$-algebra A is continuous iff $A^2 (=\operatorname{span} A \cdot A)$ is closed and of finite co-dimension. What is an analogous theorem for automatic representability? This result implies that $(l^p)^2$ is closed. Since $(l^p \cdot l^p)^- = l^p$, $(l^p)^2$ is dense in l^p. Thus $l^p = (l^p)^2$. This illustrates that in a commutative Banach $*$-algebra A, the condition $A^2 = A$ need not imply automatic representability; though it does imply automatic continuity of positive functionals [4, Theorem 37.14, p. 201].

(v) Let f be a positive functional on l^p, $f = f_a$. Let $b = (a_1, \ldots, a_n, 0, 0, \ldots,)$. Then f_b is representable and $f \geq f_b$. This shows that Corollary 3.3 fails if the assumption (c2) (and so (b2) and (a2)) are omitted. Thus, [4, Theorem 37.13, p. 200] does not hold if 'continuity' is replaced by 'representability'.

(vi) Every continuous hermitian linear functional on l^p is a difference of two positive linear functionals; though it need not be a difference of two representable positive functionals. This illustrates the crucial role of representability in Grothendieck's well-known dual characterization of C^*-algebras.

3.11. Here are some concrete examples to which Corollary 3.7 applies. (a) For a locally compact abelian group G, take $A = L^1(G)$, let $1 < p < \infty$. Take $B_1 = L^1(g) \cap L^p(G)$ with norm $\|f\|_{B_1} = \|f\|_1 + \|f\|_p$; $B_2 = \{f \in L^1(G) : \hat{f} \in L^p(\hat{G})\}$, $\|f\|_{B_2} = \|f\|_1 + \|\hat{f}\|_p$. Then each B_i is a convolution Segal algebra in $L^1(G)$ and having involution $f^*(s) = \overline{f(-s)}$. (b) Take $A = L^1(\mathbb{R}), B_k = \{f \in L^1(\mathbb{R}) \cap C^k(\mathbb{R}) |$ the kth derivative $f^{(k)} \in L^1(\mathbb{R})\}$, $\|f\|_{B_k} = \|f\|_1 + \|f^{(k)}\|_1$. By Corollaries 3.7, 3.8 and 3.9, Stinespring representable maps on $B(=$ any of above $B_i)$ are precisely those that are determined by semispectral measures on the dual group $\hat{G}$.

3.12. Another abstract Segal $*$-algebra is this. Let $A = K(H)$, C^*-algebra of all compact operators on a separable Hilbert space H, let $B = C^p(H) = \{x \in K(H) : \|x\|_p = $ [trace

$(x^*x)^{p/2}]^{1/p} < \infty\}$, $1 \le p < \infty$, the Banach $*$-algebra with norm $\|\cdot\|_p$ of von Neumann–Schatten class of operators, which is a dense $*$-ideal in A containing all finite rank operators. Let (ξ_n) be an orthonormal basis in H. Then $(C^2(H), \|\cdot\|_2)$ is a Hilbert space with norm $\|x\|_2 = \langle x,x\rangle^{1/2}$, where $\langle x,y\rangle = \sum_n \langle x\xi_n, y\xi_n\rangle$. By using [23, ch. IV, Theorem 5.5, p. 228], every positive linear functional f on $C^p(H)$ is $\|\cdot\|_p$-continuous; and by the well-known duality $\langle C^p(H), C^q(H)\rangle$, $1/p + 1/q = 1$, $\langle x,y\rangle = \text{trace } xy^*$, is of the form $f = f_a, f_a(x) = \langle x,a\rangle$ with $a \ge 0$ in $B(H)$ [22]. Further, f_a is representable iff $a \in C^1(H)$ ($=$ trace class operators) iff f is the restriction of a normal positive functional on $B(H)$. An appeal to Grothendieck's result refered to in (3.10) implies that not every positive functional is representable. Let $\phi \in CP(C^p(H), K)$ for some Hilbert space K. Then ϕ is Stinespring representable iff ϕ extends as a completely positive map $\tilde{\phi} : K(H) \to B(K)$ iff ϕ is the restriction of a normal completely positive map on $B(H)$.

3.13. Corollary 3.6 applies to smooth subalgebras of a C^*-algebra. A smooth subalgebra B of a C^*-algebra A is a dense $*$-subalgebra of A which is a Banach algebra with some norm and is spectrally invariant in A. Then $C^*(B) = A$. If B has BAI, then every completely positive map on B extends to a completely positive map on A. This, in particular, applies to the Banach $*$-algebra $C^m(A)$ of C^m-elements (and to the Frechet algebra $C^\infty(A)$ of C^∞-elements) of an action of a Lie group on a C^*-algebra A.

3.14. Banach $*$-algebras to which Theorems 2.1, 2.2 apply specifically are those that do not admit BAI. This includes non-factorizable Banach $*$-algebras [8], the algebra $R(G)$ which is linear span of positive definite functions on a compact group G, and the Fourier algebra of a locally compact, non-compact group [19]. Let G be a compact Lie group. The convolution Banach algebras $C^m(G)(C^m$-functions on G, $0 \le m < \infty)$ and $L^p(G)$, $1 < p < \infty$ are non-unital, and not admitting BAI [15, (34.40) (b), p. 357]. They are Banach $*$-algebras with involution $f^*(g) = \overline{f(g^{-1})}$ [14, Theorem 15.14, p. 197]. A map $\phi \in CP(C^m(G), H)$ (or $\phi \in CP(L^p(G), H)$) is Stinespring representable iff it extends as a completely positive map on $L^1(G)$ iff it extends as a completely positive map on the group C^*-algebra $C^*(G)$ of G iff it is determined by a completely positive definite function on G [9, p. 20]. Note that continuous positive functionals on $C^m(G)$ are given by distributions of the positive type of order m on G; whereas representable functionals are given by continuous positive definite functions on G. A similar assertion for $L^p(G)$ explains [15, (34.42 (b)), p. 358].

In particular, consider $G = T = \{z \in \mathbb{C} | \, |z| = 1\}$. Recall [16] that an orthogonal basis in a Banach algebra A is a basis $(e_n)_0^\infty$ such that $e_n e_m = \delta_{nm} e_n$. A Banach algebra with an orthogonal basis is commutative, non-unital (if infinite dimensional) and $\mathfrak{M}(A) = \{e_n^*\} \approx \mathbb{N}$, e_n^* is the coefficient functional $e_n^*(x) = \alpha_n$, where $x = \sum_0^\infty \alpha_n e_n$ is the expansion of x in (e_n). The convolution algebra $L^p(T)$, $1 < p < \infty$, admits the sequence of trigonometric polynomials $e_n(t) = t^n, n \ge 1$, as orthogonal basis, the Fourier series $f(e^{i\theta}) \sim \sum a_n e^{in\theta}$, $a_n = \hat{f}(n)$, provides expansion $f(e^{i\theta}) = \sum \hat{f}(n) e^{in\theta}$ in $\|\cdot\|_p$. It is a Banach $*$-algebra with involution $f^*(e^{i\theta}) = \overline{f(e^{-i\theta})}$. Now let $U = \{z \in \mathbb{C} | \, |z| < 1\}$. The Hardy space $H^p(U)$ is also a Banach $*$-algebra with Hadamard product $(f * g)(x) = (1/2\pi i) \int_{|z|=r} f(z) \; g(xz^{-1}) z^{-1} dz$, $|x| < r < 1$, having involution $f^*(z) = \overline{f(\bar{z})}$. The sequence $e_n(z) = z^n$, $n \in \mathbb{N}$, is an orthogonal basis for $H^p(U)$ [16], the Taylor series $f(z) = \sum_0^\infty (f^{(n)}(0)/n!) z^n$ being expansion of f in terms of (e_n). Now, by [13, ch. II, §3, p. 59]. via the radial limit $f(z) \to f(e^{i\theta})$, $H^p(U)$ is isometric to a closed subspace K of $L^p(T)$, where $K =$ the space of boundary functions of $H^p(U) =$ the closure in $L^p(T)$ of analytic polynomials. The Fourier series $f(e^{i\theta}) \sim \sum a_n e^{in\theta}$ of $f \in K$ is supported on

non-negative integers; and the Fourier coefficients $a_n = f^{(n)}(0)/n! =$ Taylor coefficients of the H^p-function $f(z) = \sum a_n z^n$. Thus the embedding preserves the multiplication and the envolution. The Banach $*$-algebras $H^p(U)$ and $L^p(T)$ are hermitian; and their enveloping C^*-algebras are $C^*(H^p(U)) \approx c_0(\mathbb{N})$ as in [3, Proposition 5.3], $C^*(L^p(T)) = C^*L^1(T))$ $= C^*(T) (=$ group C^*-algebra of $T) = C_0(\hat{T}) = C_0(\mathbb{Z})$. Thus $C^*(H^p(U)) \to C^*(L^p(T))$ injectively. Let H be a Hilbert space and $\phi : H^p(U) \to B(H)$ be positive linear. Then ϕ is extendable iff ϕ is completely positive Stinespring representable iff there exists a sequence $F(n), n \in \mathbb{N}$, of positive operators in $B(H)$ such that $\phi(f) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} F(n)$.

Acknowledgements

A careful reading of the first version of the manuscript by the referee led to the detection of a flaw in the proof of Theorem 2.1. Besides, the referee has also made suggestions regarding organization of the paper. The author sincerely thanks the referee for all these.

References

[1] Arveson W, Subalgebras of C^*-algebras, *Acta. Math.* **123** (1969) 141–224
[2] Barnes B A, The properties $*$-regularity and uniqueness of C^*-norm on general $*$-algebras, *Trans. Am. Math. Soc.* **279(2)** (1983) 841–859
[3] Bhatt S J and Karia D J, Topological algebras with C^*-enveloping algebras, *Proc. Indian Acad. Sci.* **102** (1992) 201–215
[4] Bonsall F F and Duncan J, *Complete Normed Algebras* (Springer-Verlag, Berlin, Heidelberg, New York) (1973)
[5] Bonsall F F and Duncan J, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, London Math. Soc. Lecture Notes 2, Cambridge Univ. Press) (1971)
[6] Dixon P G, Automatic continuity of positive functionals on topological involution algebras, *Bull. Australian Math. Soc.* **23** (1981) 265–281
[7] Dixmier J, C^*-algebras (North Holland Publ. Co. Amsterdam) (1977)
[8] Doran R S and Wichmann J, *Approximate Identities and Factorizations in Banach modules*, Lecture Note in Maths **768** (Springer-Verlag, Berlin Heidelberg, New York) (1979)
[9] Evans D E and Lewis J T, *Dilations of Irreversible Evolutions in Algebraic Quantum Field Theory* (Dublin Inst. Adv. Studies) (1977)
[10] Evans D E, Unbounded completely positive linear maps on C^*-algebras, *Pacific J. Math.* **66** (1976) 325–346
[11] Fell J M G and Doran R S, *Representations of $*$-algebras, locally compact groups, and Banach algebraic bundles*, (Academic Press) (1988) vol. I
[12] Fragoulopoulou M, Automatic continuity of $*$-homomorphisms on non-normed topological algebras, *Pacific J. Math.* **14** (1991) 57–70
[13] Garnet J B, *Bounded Analytic Functions* (Academic Press) (1981)
[14] Hewitt E and Ross K A, *Abstract Harmomic Analysis* (Springer-Verlag, Berlin, Gottingen Heidelberg) (1963) vol. 1
[15] Hewitt E and Ross K A, *Abstract Harmomic Analysis* (Springer-Verlag, Berlin, Gottingen Heidelberg (1970) vol. 11
[16] Husain T and Watson S, Topological algebras with orthogonal Schauder basis, *Pacific J. Math.* **91** (1980) 339–347
[17] Kadison R V and Ringrose J, *Fundamentals of Theory of Operator Algebras* (Academic Press) (1986) vol. 2
[18] Gil de Lamadrid J, Extending positive definite linear forms, *Proc. Am. Math. Soc.* **91** (1984) 593–594
[19] Leptin H, Fourier algebras of non-amenable groups, *C.R. Acad. Sci. Ser. A* **260** (1968) 1180–1181
[20] Mallios A, *Topological Algebras: Selected Topics* (Amsterdam: North-Holland) (1986)
[21] Rickart C E, *General Theory of Banach Algebras* (D Van Nostrand) (1960)

[22] Ringros J R, *Compact Non-selfadjoint Operators* (van Nostrand Renhold Math. Studies 35, van Nostrand Reinhold Publ. Co.) (1971)
[23] Schaefer H H, *Topological Vector Spaces* (MacMillan) (1964)
[24] Sebestyn Z, On representability of linear functionals on algebras, *Period. Math. Hunger.* **15** (1984) 233–239
[25] Shirali S, Representability of positive functionals, *J. London Math. Soc.* **2** (1971) 145–150
[26] Suciu I, *Function Algebras* (Noordhoft, Leyden) (1975)
[27] Takesaki M, *Theory of Operator Algebras* (Springer-Verlag, Berlin, Heidelberg, New York) (1979)
[28] Weidmann J, *Linear Operators in Hilbert Spaces* (Springer-Verlag, Berlin, Heidelberg, New York) (1980)

SUBJECT INDEX

AUTHOR INDEX

Proceedings of the Indian Academy of Sciences
Mathematical Sciences

Volume 108
1998

Published by the Indian Academy of Sciences
Bangalore 560 080

Proceedings of the Indian Academy of Sciences

(Mathematical Sciences)

Subscription Rates

All countries except India................. $100
(Price includes AIR MAIL charges)

India....................................... Rs 150

Annual subscriptions are available for **Individuals** for India and abroad at the concessional rates of Rs. 75/- and $30 respectively.

All correspondence regarding subscription should be addressed to **The Circulation Department** of the Academy.

Editorial Office

Indian Academy of Sciences, C V Raman Avenue,
P.B. No. 8005, Bangalore 560 080, India

Telephone: 80-334 2546
Telefax: 91-80-334 6094
Website: http://www.ias.ernet.in

Email: mathsci@ias.ernet.in

Proceedings of the Indian Academy of Sciences

Mathematical Sciences

Volume 108, 1998

VOLUME CONTENTS

Number 1, February 1998

Number 2, June 1998

Number 3, October 1998

Proceedings of the Indian Academy of Sciences
Mathematical Sciences

Notes on the preparation of papers

Authors wishing to have papers published in the *Proceedings* should send them to

> The Editor, Proceedings (Mathematical Sciences), Indian Academy of Sciences, C V Raman Avenue, P B No. 8005, Bangalore 560 080, India

OR

> Prof. S G Dani, School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India

Three copies of the paper must be submitted.

The papers must normally present results of original work. Critical reviews of important fields will also be considered for publication. Submission of the typescript will be held to imply that it has not been previously published, it is not under consideration for publication elsewhere and that, if accepted, it will not be published elsewhere. A paper in applied areas will be considered only on the basis of its mathematical content.

Typescript

Papers should be typed double spaced with ample margin on all sides on white bond paper of size 280 × 215 mm. This also applies to the abstract, tables, figure captions and the list of references which are to be typed on separate sheets.

Title page:

(1) The title of the paper must be short and contain words useful for indexing.
(2) The authors' names should be followed by the names and addresses of the institutions of affiliation and email address.
(3) An *abbreviated running title* of not more than 50 letters and spaces should be given.

Abstract:

Each paper must be accompanied by an abstract describing, in not more than 200 words, the significant results reported in the paper.

Keywords:

Between 3 and 6 keywords must be provided for indexing and information retrieval.

The Text:

The paper should preferably start with an introduction in which the results are placed in perspective and some indication of the methods of proof is given.

1. *Markings*: The copy intended for the printer, should be marked appropriately to make it unambiguous. The following conventions may be followed for the purpose of indicating special characters (A list of special characters is included at the end):

a) Indicating characters by underlining

Italics	Single underline
Bold face	Wavy underline
Greek	Blue underline
Fraktur (Gothic)	Red underline
Script	Green underline
Open face	Brown underline

All parameters in equations are normally printed in italics and numerals in upright typeface.

b) In doubtful cases the position of superscripts and subscripts (exponents and indices) should be clearly marked, for example b_{j}, a^{i}, $b^{a^{x}}$, $b_{x_{i}}$, etc.

c) Formulae extending beyond the printed line will be broken by the typesetter at an appropriate place; to avoid any expensive alterations at the proof stage the author may indicate in the manuscript where the formulae should be broken.

d) Various likely ambiguous spots may be explained by pencilled notes in the margin. The following symbols are frequently confused and need to be clarified.

$$^{\circ}, o, O, 0; \cup, \bigcup, U; \times, x, X, \chi; k, \kappa; \mathrm{v}, \nu, \upsilon; \theta, \Theta, \phi, \varphi, \Phi, \varnothing; \psi, \Psi; \varepsilon, \in;$$

a, α, $\propto$; B, β; r, γ; σ, 6; $+$, $\dagger$; i, $\imath$; a', a^1; the symbol a and the indefinite article a; also the handwritten Roman letters,
$c, C; e, l; I, J; k, K; o, O; p, P; s, S; u, U; v, V; w, W; x, X; z, Z.$

2. *Notation and style.* Notation and style should be chosen carefully, keeping the printing aspect in mind. The following table indicates preferred forms for various mathematical usages.

preferred form	instead of	preferred form	instead of
$A^*, \tilde{b}, \gamma', \mathbf{v}$, etc.	$\bar{A}, \hat{b}, \hat{\gamma}, \vec{v}$, etc.	$\exp(-(x^2+y^2)/a^2)$	$e^{-((x^2+y^2)/a^2)}$
lim sup, proj lim	$\overline{\lim}$, $\underline{\lim}$		
$f: A \to B$	$A \xrightarrow{f} B$		
$\Sigma_{n=1}^{\infty}$	$\sum\limits_{n=1}^{\infty}$	$\dfrac{\cos(1/x)}{(a+b/x)^{1/2}}$	$\dfrac{\cos\frac{1}{x}}{\sqrt{a+\frac{b}{x}}}$

Tables:

All tables must be numbered consecutively in arabic numerals in the order of appearance in the text. The tables should be self-contained and must have a descriptive title.

Figures:

A figure should be included only when it would be substantially helpful to the reader in understanding the subject matter. The figures should be numbered consecutively in arabic numerals in the order of appearance in the text and the location of each figure should be clearly indicated in margin as "Figure 1 here". The figure captions must be typed on a separate sheet.

Line drawings must be in Indian ink on good quality tracing paper. Lines must be drawn sufficiently thick for reduction to a half or third of the original size (0·3 mm for axes and 0·6 mm for curves are suggested).

List of symbols:

For the convenience of printers, authors should attach to the manuscript a complete list of symbols used in the manuscript identified typographically.

References

References should be cited in the text by serial numbers only (e.g. [3]). They should be listed alphabetically by the author's name (the first author's name in the case of joint authorship) at the end of the paper. In describing each reference the following order should be observed: the author's name followed by initials, title of the article, the name of the journal, the volume number, the year and the page numbers. Standard abbreviations of journal titles should be used. It would be worthwhile to cross-check all references cited in the text with the ones given in the list at the end.

A typical reference to an article in a journal would be like:

[1] Narasimhan M S and Ramanan S, Moduli of vector bundles on a compact Riemann surface, *Ann. Math.* **89** (1969) 14–51

A reference to a book would be on the following lines:

[2] Royden H, Invariant metrics on Teichmüller space, in: Contributions to analysis (eds) L Ahlfors *et al* (1974) (New York: Academic Press) pp. 393–399

Footnotes:

Footnotes to the text should be avoided if possible. If unavoidable, they should be numbered consecutively, and typed on a separate sheet.

Proofs:

The journal is now typeset by computer photocomposition and only a page-proof is supplied to authors. While the process yields better results, it makes corrections at proof stage very difficult and expensive. Deleting a letter or word in the proof can mean resetting the whole line or paragraph. Similarly, addition of a sentence or paragraph might lead to resetting the whole page and sometimes even the whole article.

Authors are requested to prepare the manuscript carefully before submitting it for publication to minimize corrections and alterations in the proof stage which increase publication costs. The proofs sent to the authors together with the reprint order form must be returned to the editorial office *within two days of their receipt.*

Reprints:

50 reprints of each article will be supplied free of charge.

LATIN LIGHT ROMAN GROUP 1

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz 1234567890&
.,:;!?''-—·*/%()[]£$†‡°ı

MATH SIGNS

⊂ ⊃ × ≃ ∞ ∝ ∈ ∉ □ ○ √ √ ⇌ ⇒ ⇐ ≑ ≠ ···
⊥ < > ⊏ ⊕ ⊗ ⊞ ← ⊙ ↑ ∧ ∨ ⇝ ¬
∟ ⊣ ♭ → ⌉ ⊊ + − ± × = ÷ ~ → ← ≡ ′ ″ \
()[]⟨⟩{}∶∴ ⩽ ⩾ ≦ ≧ ⊆ ⊇ ∩ ∪ ⋂ ⋃ < > ≮ ≯
ℏ ∇ ∃ ≅ ≿ ≾ ⊇ ⊆ ≷ ≶ ∓ ⊕ ⊖ ⊗ ≏ ≺ ≻ ⋚ ⋛
↔ ⇔ ≘ ≳ ≲ ⊂ ⊃ ⊥ ∧ ∨ ℸ ℵ ℶ ⋞ ⋟ ⋜ ⋝ ↕ ⩽ ⩾ ∮
≪ ≫ ⇆ ⇄ ∅ ♮ ⇋ ⇌ ⊧ ↖ ↘ ↗ ↙ ‰ § ¶

ASTRONOMICAL SIGNS

☿ ♃ ♄ ♌ ♈ ♉ ♍ ♏ ☊ ♂ ♀ Ð Þ þ

LATIN LIGHT ITALIC GROUP 1

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz 1234567890&
.,:;!?''-—·/%()[]£$†‡°* ı

LATIN BOLD ROMAN GROUP 2

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz 1234567890&
.,:;!?''-—·*/%()[]£$ı

LATIN BOLD ITALIC GROUP 2

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz&
.,:;!?''-—·*/%()[]£$ı

LATIN LIGHT ROMAN GROUP 7

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz 1234567890
.·/()[]±‡‡ min max lim log

LATIN LIGHT ITALIC GROUP 7

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz

MATH SIGNS GROUP 7

⊂ ⊃ × ≃ ∞ ∝ ∈ ∉ □ ○ √ √ ⇌ ⇒ ⇐ ≑ ≠ ···
⊥ < > ⊏ ⊕ ⊗ ⊞ ← ⊙ ↑ ∧ ∨ ⇝ ¬
∟ ⊣ ♭ → ⌉ ⊊ + − ± × = ÷ ~ → ← ≡ ′ ″ \
()[]⟨⟩{}∶∴ ⩽ ⩾ ≦ ≧ ⊆ ⊇ ∩ ∪ ⋂ ⋃ < > ≮ ≯
ℏ ∇ ∃ ≅ ≿ ≾ ⊇ ⊆ ≷ ≶ ∓ ⊕ ⊖ ⊗ ≏ ≺ ≻ ⋚ ⋛
↔ ⇔ ≘ ≳ ≲ ⊂ ⊃ ⊥ ∧ ∨ ℸ ℵ ℶ ⋞ ⋟ ⋜ ⋝ ↕ ⩽ ⩾ ∮
≪ ≫ ⇆ ⇄ ∅ ♮ ⇋ ⇌ ⊧

GREEK LIGHT UPRIGHT GROUP 3

ΓΛΘΔΞΠΣΦΨΩ∇∑∏∫ΠΣ
αβγδεζηθκλμνξπρστφχψωφ∂ϑϖυκϱε

GREEK LIGHT INCLINED GROUP 4

ΓΛΘΔΞΠΣΦΨΩΔ
αβγδεζηθκλμνξπρστφχψωφ

GREEK BOLD UPRIGHT GROUP 5

ΓΔΘΞΛΠΣΥΦΨΩ∇ςεςε
αβγδεζηθκλμνξπρστφχψωφϑϖυςϱι

GREEK BOLD INCLINED GROUP 6

ΓΛΘΞΛΠΣΥΦΨΩςεςε
αβγδεζηθκλμνξπρστφχψωφϑϖυςϱι

GREEK LIGHT UPRIGHT GROUP 8

ΓΛΘΛΞΠΣΦΨΩ∇∫αβγδεζηθκλμνξπρστφχψωφ

GREEK LIGHT INCLINED GROUP 9

ΓΔΘΛΞΠΣΦΨΩαβγδεζηθκλμνξπρστφχψωφ

LATIN LIGHT ITALIC GROUP 10

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz

LATIN LIGHT ROMAN GROUP 10

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz 1234567890
()[]∪∩⊕⊖·⊗∅∧∨*√

LATIN LIGHT SCRIPT GROUP 18

𝒜ℬ𝒞𝒟ℰℱ𝒢ℋℐ𝒥𝒦ℒℳ𝒩𝒪𝒫𝒬
ℛ𝒮𝒯𝒰𝒱𝒲𝒳𝒴𝒵
𝒶𝒷𝒸𝒹ℯ𝒻ℊ𝒽𝒾𝒿𝓀𝓁𝓂𝓃ℴ𝓅𝓆𝓇𝓈𝓉𝓊𝓋𝓌𝓍𝓎𝓏

LATIN BOLD SCRIPT 20

𝓐𝓑𝓒𝓓𝓔𝓕𝓖𝓗𝓘𝓙𝓚𝓛𝓜𝓝𝓞𝓟𝓠
𝓡𝓢𝓣𝓤𝓥𝓦𝓧𝓨𝓩
𝓪𝓫𝓬𝓭𝓮𝓯𝓰𝓱𝓲𝓳𝓴𝓵𝓶𝓷𝓸𝓹𝓺𝓻𝓼𝓽𝓾𝓿𝔀𝔁𝔂𝔃

FRACTURE LIGHT GROUP 22

𝔄𝔅ℭ𝔇𝔈𝔉𝔊ℌℑ𝔍𝔎𝔏𝔐𝔑𝔒𝔓𝔔ℜ𝔖𝔗𝔘𝔙𝔚𝔛𝔜ℨ
𝔞𝔟𝔠𝔡𝔢𝔣𝔤𝔥𝔦𝔧𝔨𝔩𝔪𝔫𝔬𝔭𝔮𝔯𝔰𝔱𝔲𝔳𝔴𝔵𝔶𝔷

FRACTURE BOLD GROUP 23

𝕬𝕭𝕮𝕯𝕰𝕱𝕲𝕳𝕴𝕵𝕶𝕷𝕸𝕹𝕺𝕻𝕼𝕽𝕾𝕿𝖀𝖁𝖂𝖃𝖄𝖅
𝖆𝖇𝖈𝖉𝖊𝖋𝖌𝖍𝖎𝖏𝖐𝖑𝖒𝖓𝖔𝖕𝖖𝖗𝖘𝖙𝖚𝖛𝖜𝖝𝖞𝖟

LATIN OPEN FACE GROUP 35

𝔸𝔹ℂ𝔻𝔼𝔽𝔾ℍ𝕀𝕁𝕂𝕃𝕄ℕ𝕆ℙℚℝ𝕊𝕋𝕌𝕍𝕎𝕏𝕐ℤ
𝕘𝕙𝕜𝕤𝕥𝟙⦅⦆

LARGE DISPLAY CHARACTERS

()[]{}⟨⟩[]/√∮‖

PHONETICS SYMBOLS

ąɑɒqƀɔɕχdðɗɖ
ʤʣęɛɜɘɟɠɦɧɥɦɨɫɬɮɱɰɯŋɲɳɸøʊɸ
ɋʁɹɾβʂʃʅɵθʧʃŋɪʉʌʋɣɣʎʏʐʑʒ ʔʕ : ʻ ʼ ¿¡